Robust Hamiltonicity of Dirac graphs

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Abstract

A graph is Hamiltonian if it contains a cycle which passes through every vertex of the graph exactly once. A classical theorem of Dirac from 1952 asserts that every graph on n vertices with minimum degree at least $n/2$ is Hamiltonian. We refer to such graphs as Dirac graphs. In this paper we extend Dirac's theorem in two directions and show that Dirac graphs are robustly Hamiltonian in a very strong sense. First, we consider a random subgraph of a Dirac graph obtained by taking each edge independently with probability p, and prove that there exists a constant C such that if $p \geq C \log n / n$, then a.a.s. the resulting random subgraph is still Hamiltonian. Second, we prove that if a $(1 : b)$ Maker-Breaker game is played on a Dirac graph, then Maker can construct a Hamiltonian subgraph as long as the bias b is at most $cn/\log n$ for some absolute constant $c > 0$. Both of these results are tight up to a constant factor, and are proved under one general framework.

1 Introduction

A *Hamilton cycle* of a graph is a cycle which passes through every vertex of the graph exactly once, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity is one of the most central notions in Graph Theory, and has been intensively studied by numerous researchers. The problem of deciding Hamiltonicity of a graph is one of the NP-complete problems that Karp listed in his seminal paper [20], and accordingly, one cannot hope for a simple classification of such graphs. Still, there are many results deriving properties sufficient for Hamiltonicity. For example, a classical result proved by Dirac in 1952 (see, e.g., [17, Theorem 10.1.1]), asserts that every graph on n vertices of minimum degree at least $\frac{n}{2}$ is Hamiltonian. In this context, we say that a graph is a *Dirac graph* if it has minimum degree at least $\frac{n}{2}$. Dirac's theorem is one of the most influential results in the study of Hamiltonicity and by now there are many related known results (see, e.g., [13]).

Let G be a graph and $\mathcal{P}$ be a graph property. Many results in Graph Theory state that "under certain conditions, G has property $\mathcal{P}$". Once such a result is established, it is natural to ask: "How

strongly does G possess $\mathcal{P}$?". In other words, we want to determine the robustness of G with respect to $\mathcal{P}$. Recently, such questions were extensively studied by many researchers.

In order to answer the question about robustness, we would need some kind of a measure of this phenomenon. There are several measures of robustness that were proposed so far. For example, one can measure the robustness of Dirac graphs with respect to Hamiltonicity by computing the number of Hamilton cycles that a Dirac graph must contain. Indeed, confirming a conjecture of Sárközy, Selkow, and Szemerédi [31], Cuckler and Kahn [15] proved that every Dirac graph contains at least $n!/(2+o(1))^n$ Hamilton cycles. Another measure is the so called *resilience*, whose systematic study was initiated by Sudakov and Vu [32], and has been intensively studied afterwards, see, e.g., [2, 4, 9, 16, 25] and their references (resilience is closely related to the notion of fault tolerance, see, e.g., [3]). Roughly speaking, for monotone increasing graph properties, these measures compute the robustness in terms of the number of edges one must delete from G locally or globally in order to destroy the property $\mathcal{P}$. In this paper, we would like to revisit Dirac's theorem and study different settings which can be used to demonstrate its robustness. Our main results show how to strengthen Dirac's theorem in two ways.

1.1 Random subgraph

Let $G(n, p)$ be the binomial model of random graphs, which denotes the probability space whose points are graphs with vertex set $[n] = \{1, \ldots, n\}$ where each pair of vertices forms an edge randomly and independently with probability p. We say that $G(n, p)$ possesses a graph property $\mathcal{P}$ *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $G(n, p)$ possesses $\mathcal{P}$ tends to 1 as n goes to infinity. The earlier results on Hamiltonicity of random graphs were proved by Pósa [29], and Korshunov [23]. Improving on these results, Bollobás [10], and Komlós and Szemerédi [22] independently proved that if $p \geq \frac{\log n + \log \log n + \omega(n)}{n}$ for some function $\omega(n)$ that goes to infinity as n goes to infinity, then $G(n, p)$ is a.a.s. Hamiltonian. The range of p cannot be improved, since if $p \leq \frac{\log n + \log \log n - \omega(n)}{n}$, then $G(n, p)$ a.a.s. has a vertex of degree at most one.

An equivalent way of describing $G(n, p)$ is as the probability space of graphs obtained by taking every edge of the complete graph K_n independently with probability p. A variety of questions can be asked when we consider a host graph G other than K_n, and consider the probability space of graphs obtained by taking every edge of it independently with probability p (we denote this probability space as G_p).

The following question can be placed in this context and can be also viewed as an attempt to understand the robustness of Dirac's theorem. Let G be a graph of minimum degree at least $\frac{n}{2}$ and note that G is Hamiltonian by Dirac's theorem. Since Hamiltonicity is a monotone graph property, we know that there exists a threshold p_0 (see [12]) such that if $p \gg p_0$, then G_p is a.a.s. Hamiltonian, and if $p \ll p_0$, then G_p is a.a.s. not Hamiltonian. For random graphs, the threshold for Hamiltonicity is $p_0 = \frac{\log n}{n}$ (it is moreover a sharp threshold). What is the Hamiltonicity threshold for G_p, in particular, does G_p stay Hamiltonian for $p \ll 1$? Our main theorem provides an answer to this question.

Theorem 1.1. *There exists a positive constant C such that for $p \geq \frac{C \log n}{n}$ and a graph G on n vertices of minimum degree at least $\frac{n}{2}$, the random subgraph G_p is a.a.s. Hamiltonian.*

This theorem establishes the correct order of magnitude of the threshold function since if $p \leq (1 + o(1))\frac{\log n}{n}$, then the graph a.a.s. has isolated vertices. Also, since there are graphs with minimum degree $\frac{n}{2} - 1$ which are not even connected, the minimum degree condition cannot be improved. Moreover, our theorem can actually be viewed as an extension of Dirac's theorem since the case $p = 1$ is equivalent to Dirac's theorem.

1.2 Hamiltonicity game

Let V be a set of elements and $\mathcal{F} \subseteq 2^V$ be a family of subsets of V. A *Maker-Breaker game* involves two players, named Maker and Breaker respectively, who alternately occupy the elements of V, the *board* of the game. The game ends when there are no unoccupied elements of V. Maker wins the game if in the end, the vertices occupied by Maker contain as a subset at least one of the sets in $\mathcal{F}$, the family of *winning sets* of the game. Breaker wins otherwise.

Chvátal and Erdős [14] were the first to consider biased Maker-Breaker games on the edge set of the complete graph. They realized that natural graph games are often "easily" won by Maker when played fairly (that is when Maker and Breaker each claim one element at a time). Thus for many graph games, it is natural to give Breaker some advantage. In a $(1 : b)$ Maker-Breaker game we follow the same rule as above, but Maker claims one element each round while Breaker claims b elements each round. It is not too difficult to see that Maker-Breaker games are *bias monotone*. More specifically, if for some fixed game, Maker can win the $(1 : b)$ game, then Maker can win the $(1 : b')$ game for every $b' < b$. Thus it is natural to consider the *critical bias* of a game, which is defined as the maximum b_0 such that Maker wins the $(1 : b_0)$ game.

One of the first biased games that Chvátal and Erdős considered in their paper was the Hamiltonicity game played on the edge set of the complete graph. They proved that the $(1 : 1)$ game is Maker's win, and that for any fixed positive ε and $b(n) \geq (1 + \varepsilon)\frac{n}{\log n}$, the $(1 : b)$ game is Breaker's win for large enough n. They then conjectured that the critical bias of this game should go to infinity as n goes to infinity. Bollobás and Papaioannou [11] verified their conjecture and proved that the critical bias is at least $c\frac{\log n}{\log \log n}$ for some constant $c > 0$. Beck [6] improved on this result by proving that the critical bias is at least $(\frac{\log 2}{27} - o(1))\frac{n}{\log n}$, thereby establishing the correct order of magnitude of the critical bias. Krivelevich and Szabó [26] further improved this result, and recently Krivelevich [24] established the fact that the critical bias of this game is asymptotically $\frac{n}{\log n}$. We refer the reader to [8] for more information on Maker-Breaker games, and general positional games.

In this context, and similarly to that of the question considered in the previous subsection, we would like to strengthen Dirac's theorem from the Maker-Breaker game point of view. Let G be a graph of minimum degree at least $\frac{n}{2}$ and consider the Hamiltonicity Maker-Breaker game played on G (note that G is Hamiltonian by Dirac's theorem). We can then ask the following questions: "will Maker win the $(1 : 1)$ game on any Dirac graph?", and if so, then "what is the largest bias b such that Maker wins the $(1 : b)$ game?". In this paper we establish the threshold b_0 such that if $b \ll b_0$, then Maker wins, and if $b \gg b_0$, then Breaker wins.

Theorem 1.2. *There exists a constant $c > 0$ such that for $b \leq \frac{cn}{\log n}$ and a graph G on n vertices of minimum degree at least $\frac{n}{2}$, Maker has a winning strategy for the $(1 : b)$ Maker-Breaker Hamiltonicity game on G.*

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Our theorem implies that the critical bias of this game has order of magnitude $\frac{n}{\log n}$ (note that the critical bias is at most $(1+o(1))\frac{n}{\log n}$ by the result of Chvátal and Erdős mentioned above). Note that in this theorem, once all the elements of the board are claimed, the edge density of Maker's graph is of order of magnitude $\frac{\log n}{n}$ and this is the same as in Theorem 1.1. This suggests that as in many other Maker-Breaker games, the "probabilistic intuition", a relation between the critical bias and the threshold probability of random graphs, holds here as well (see, [5, 7]). In fact, this is not a coincidence, and we will prove both theorems under one unified framework.

Notation. A graph $G = (V, E)$ is given by a pair of its vertex set $V = V(G)$ and edge set $E = E(G)$. We use $|G|$ or $|V|$ to denote the size of its vertex set. For a subset X of vertices, we use $e(X)$ to denote the number of edges within X, and for two sets X, Y, we use $e(X, Y)$ to denote the number of pairs (x, y) such that $x \in X, y \in Y$, and $\{x, y\}$ is an edge (note that $e(X, X) = 2e(X)$). $G[X]$ denotes the subgraph of G induced by a subset of vertices X. We use $\overline{X}$ to denote the complement $V \setminus X$ of X, and $N(X)$ to denote the collection of vertices which are adjacent to some vertex of X. For two graphs G_1 and G_2 over the same vertex set V, we define their intersection as $G_1 \cap G_2 = (V, E(G_1) \cap E(G_2))$, their union as $G_1 \cup G_2 = (V, E(G_1) \cup E(G_2))$, and their difference as $G_1 \setminus G_2 = (V, E(G_1) \setminus E(G_2))$.

When there are several graphs under consideration, to avoid ambiguity, we use subscripts such as $N_G(X)$ to indicate the graph that we are currently interested in. We also use subscripts with asymptotic notations to indicate dependency. For example, Ω_ε will be used to indicate that the hidden constant depends on ε. Throughout the paper, whenever we refer, for example, to a function with subscript as $f_{3.1}$, we mean the function f defined in Lemma/Theorem 3.1. To simplify the presentation, we often omit floor and ceiling signs whenever these are not crucial and make no attempts to optimize absolute constants involved. We also assume that the order n of all graphs tends to infinity and therefore is sufficiently large whenever necessary. All logarithms will be in base $e \approx 2.718$.

2 Dirac graphs

The following lemma used by Sárközy and Selkow [30], and by Cuckler and Kahn [15], classifies Dirac graphs into three categories (a similar lemma has also been used by Komlós, Sarközy, and Szemerédi [21]). This classification is an important tool in controlling Dirac graphs. A *half set* of a graph is a subset of the vertex set which has size either $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$.

Lemma 2.1. *Let $\alpha \leq \frac{1}{320}$ and $\gamma \leq \frac{1}{10}$ be fixed positive reals such that $\gamma \geq 32\alpha$. If n is large enough, then for every graph G on n vertices with minimum degree at least $\frac{n}{2}$, one of the following holds:*

(i) *$e(A, B) \geq \alpha n^2$ for all half-sets A and B (not necessarily disjoint),*

(ii) *There exists a set A of size $\frac{n}{2} \leq |A| \leq (\frac{1}{2} + 16\alpha)n$ such that $e(A, \overline{A}) \leq 6\alpha n^2$, and the induced subgraphs on both A and $\overline{A}$ have minimum degree at least $\frac{n}{5}$, or*

(iii) *There exists a set A of size $\frac{n}{2} \leq |A| \leq (\frac{1}{2} + 16\alpha)n$ such that the bipartite graph induced by the edges between A and $\overline{A}$ has at least $(\frac{1}{4} - 14\alpha)n^2$ edges, and minimum degree at least $\frac{\gamma}{2}n$. Moreover, either $|A| = \lceil \frac{n}{2} \rceil$, or the induced subgraph $G[A]$ has maximum degree at most γn.*

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Proof. Assume that property (i) does not hold, i.e., that $e(A, B) < \alpha n^2$ for some two half-sets A and B. For simplicity, we assume that n is even and $|A| = |B| = \frac{n}{2}$ (for odd n, some small order terms will be added to the computation). Note that in this case we have $|\overline{A \cup B}| = |A \cap B|$, it follows that for all $v \in A \cap B$,

$$|N(v) \cap (A \cup B)| \geq |N(v)| - |\overline{A \cup B}| \geq \frac{n}{2} - |A \cap B|.$$

Therefore,

$$\alpha n^2 > e(A, B) \geq \sum_{v \in A \cap B} |N(v) \cap (A \cup B)| \geq |A \cap B| \cdot \left(\frac{n}{2} - |A \cap B|\right).$$

If $5\alpha n \leq |A \cap B| \leq (\frac{1}{2} - 5\alpha)n$, then the right hand side of the above estimate is at least $5\alpha n \cdot (\frac{1}{2} - 5\alpha)n$, which by $\alpha \leq \frac{1}{320}$ is larger than αn^2. Thus the above inequality implies that either $|A \cap B| \leq 5\alpha n$ or $|A \cap B| \geq (\frac{1}{2} - 5\alpha)n$. We consider these two cases separately.

Case 1 : $|A \cap B| \leq 5\alpha n$.

In this case, we have $|\overline{A} \setminus B| \leq 5\alpha n$ and thus $e(A, \overline{A}) \leq e(A, B) + |\overline{A} \setminus B| \cdot |A| \leq \alpha n^2 + \frac{5}{2}\alpha n^2 \leq 4\alpha n^2$. Let $A_0 = A$ and $B_0 = \overline{A}$. Let A_1 be the set of vertices of A_0 which have at most $\frac{n}{4}$ neighbors in A_0, and note that by the minimum degree condition of the graph G, every vertex in A_1 has at least $\frac{n}{4}$ neighbors in B_0. Since $e(A, \overline{A}) \leq 4\alpha n^2$, we have $|A_1| \leq 16\alpha n$. Similarly, we can define B_1 so that $|B_1| \leq 16\alpha n$. Let $A' = (A_0 \setminus A_1) \cup B_1$ and $B' = (B_0 \setminus B_1) \cup A_1$. We obtain a new partition $V = A' \cup B'$ such that

$$\left(\frac{1}{2} - 16\alpha\right) n \leq |A'|, |B'| \leq \left(\frac{1}{2} + 16\alpha\right) n$$

and the minimum degree inside each part is at least $\frac{n}{4} - 16\alpha n \geq \frac{n}{5}$. Moreover, we have

$$
\begin{aligned}
e(A', B') &= e(A_0, B_0) - e(A_1, B_0 \setminus B_1) + e(A_1, A_0 \setminus A_1) - e(B_1, A_0 \setminus A_1) + e(B_1, B_0 \setminus B_1) \\
&\leq e(A_0, B_0) - e(A_1, B_0) + e(A_1, A_0) - e(B_1, A_0) + e(B_1, B_0) + 2e(A_1, B_1),
\end{aligned}
$$

which by $e(A_1, B_0) \geq e(A_1, A_0)$ and $e(B_1, A_0) \geq e(B_1, B_0)$ gives

$$e(A', B') \leq e(A_0, B_0) + 2e(A_1, B_1) \leq 4\alpha n^2 + 2|A_1||B_1| \leq 4\alpha n^2 + 2^9 \alpha^2 n^2 \leq 6\alpha n^2.$$

Since $|A' \cup B'| = n$, the larger set among A' and B' has size at least $\frac{n}{2}$, and it satisfies property (ii).

Case 2 : $|A \cap B| \geq (\frac{1}{2} - 5\alpha)n$.

In this case, we have $|A \setminus B| \leq 5\alpha n$, and therefore $e(A, A) \leq e(A, B) + |A \setminus B| \cdot |A| \leq \alpha n^2 + \frac{5}{2}\alpha n^2 \leq 4\alpha n^2$. By the minimum degree condition, we have $e(A, \overline{A}) \geq |A| \cdot \frac{n}{2} - e(A, A) \geq (\frac{1}{4} - 4\alpha)n^2$. Let $A_0 = A$ and $B_0 = \overline{A}$. Let A_1 be the set of vertices of A_0 which have at most $\frac{n}{4}$ neighbors in B_0. Note that by the minimum degree condition of the graph G, every vertex in A_1 has at least $\frac{n}{4}$ neighbors in A_0. Also, by the estimate

$$\left(\frac{1}{4} - 4\alpha\right) n^2 \leq e(A_0, B_0) \leq \frac{n}{4} \cdot |A_1| + \frac{n}{2} \cdot \left(\frac{n}{2} - |A_1|\right) = \frac{n^2}{4} - \frac{n}{4} \cdot |A_1|,$$

we have $|A_1| \leq 16\alpha n$. Similarly define B_1 so that $|B_1| \leq 16\alpha n$. Let $A' = (A_0 \setminus A_1) \cup B_1$ and $B' = (B_0 \setminus B_1) \cup A_1$. We have

$$
\begin{aligned}
e(A', B') &= e(A_0, B_0) - e(A_1, B_0 \setminus B_1) + e(A_1, A_0 \setminus A_1) - e(B_1, A_0 \setminus A_1) + e(B_1, B_0 \setminus B_1) \\
&\geq e(A_0, B_0) - e(A_1, B_0) + e(A_1, A_0) - e(B_1, A_0) + e(B_1, B_0) - 2e(A_1, B_1),
\end{aligned}
$$

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which by $e(A_1, B_0) \leq e(A_1, A_0)$ and $e(B_1, A_0) \leq e(B_1, B_0)$ gives

$$e(A', B') \geq e(A_0, B_0) - 2e(A_1, B_1) \geq \left(\frac{1}{4} - 4\alpha\right) n^2 - 2|A_1||B_1|$$

$$\geq \left(\frac{1}{4} - 4\alpha\right) n^2 - 2^9 \alpha^2 n^2 \geq \left(\frac{1}{4} - 6\alpha\right) n^2,$$

and all vertices in A' have at least $\frac{n}{4} - 16\alpha n \geq \frac{n}{5}$ neighbors in B' (and vice versa).

Since $|A' \cup B'| = n$, we may assume without loss of generality that $|A'| \geq \lceil \frac{n}{2} \rceil$. Note that $|A'| \leq |A_0| + |B_1| \leq \left(\frac{1}{2} + 16\alpha\right) n$. If $|A'| = \lceil \frac{n}{2} \rceil$ or $G[A']$ has maximum degree at most γn, then we have already found our set A'. Otherwise, move a vertex in A' which has at least γn neighbors in A' to the other side B', and update A', B' accordingly. Repeat this until we reach the point where $|A'| = \lceil \frac{n}{2} \rceil$ or $G[A']$ has maximum degree at most γn. Since we moved at most $16\alpha n$ vertices from A' to B', all the vertices in B' have at least $\gamma n - 16\alpha n \geq \frac{\gamma}{2} n$ neighbors in A'. On the other hand, all the remaining vertices in A' still have at least $\frac{n}{5} \geq \frac{\gamma}{2} n$ neighbors in B'. Finally, $e(A', B')$ decreases by at most $16\alpha n \cdot \frac{n}{2} = 8\alpha n^2$ and is still at least $(\frac{1}{4} - 14\alpha)n^2$. Thus we found our claimed set as in property (iii). $\qquad\square$

3 Rotation and extension

We will prove our two main theorems under one general framework provided in this section. Our main tool is Pósa's rotation-extension technique which first appeared in [29] (see also [28, Ch. 10, Problem 20]). We start by briefly discussing this powerful tool, which exploits the expansion property of the graph.

Let G be a connected graph and let $P = (v_0, \cdots, v_\ell)$ be a path on some subset of vertices of G (P is not necessarily a subgraph of G). If $\{v_0, v_\ell\}$ is an edge of the graph, then we can use it to close P into a cycle. Since G is connected, either the graph $G \cup P$ is Hamiltonian, or there exists a longer path in this graph. In the second case, we say that we *extended* the path P.

Assume that we cannot directly extend P as above, and assume that G contains an edge of the form $\{v_\ell, v_i\}$ for some i. Then $P' = (v_0, \cdots, v_i, v_\ell, v_{\ell-1}, \cdots, v_{i+1})$ forms another path of length ℓ in $G \cup P$ (see Figure 3.1). We say that P' is obtained from P by a *rotation* with *fixed endpoint* v_0, *pivot point* v_i, and *broken edge* (v_i, v_{i+1}). Note that after performing this rotation, we can now close a cycle of length ℓ also using the edge $\{v_0, v_{i+1}\}$ if it exists in $G \cup P$. As we perform more and more rotations, we will get more such candidate edges (call them *closing edges*). The rotation-extension technique is employed by repeatedly rotating the path until one can find a closing edge in the graph, thereby extending the path.
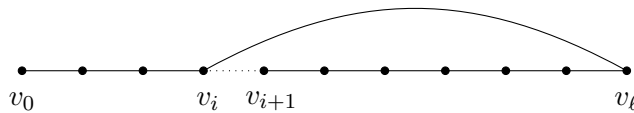

Figure 3.1: Rotating a path.

Let P'' be a path obtained from P by several rounds of rotations. An important observation that we will use later is that for every interval $I = (v_j, \cdots, v_k)$ of vertices of P ($1 \le j < k \le \ell$), if no edges of I were broken during these rotations, then I appears in P'' either exactly as it does in P, or in the reversed order. We define the *orientation*, or *direction*, of a path P'' with respect to an interval I to be *positive* in the former situation, and *negative* in the latter situation.

We will use rotations and extensions as described above to prove our main theorem. The main technical twist is to split the given graph into two graphs, where the first graph will be used to perform rotations, and the second graph to perform extensions. Similar ideas, such as *sprinkling*, have been used in proving many results on Hamiltonicity of random graphs. The one closest to our implementation appears in the recent paper of Ben-Shimon, Krivelevich, and Sudakov [9].

3.1 Rotation-Extension for general graphs

In this subsection, we develop a framework useful in tackling the first and the second cases of Lemma 2.1. We assume that all the graphs appearing in this subsection are defined over a fixed vertex set V of size n (therefore if there are several graphs under consideration, then they share the same vertex set). We first specify the roles of the graphs performing rotations and extensions.

Definition 3.1. Let ξ be a positive constant. We say that a graph G has property $\mathcal{RE}(\xi)$ if it is connected, and for every path P with a fixed edge e, (i) there exists a path containing e longer than P in the graph $G \cup P$, or (ii) there exists a set of vertices S_P of size $|S_P| \ge \xi n$ such that for every vertex $v \in S_P$, there exists a set T_v of size $|T_v| \ge \xi n$ such that for every $w \in T_v$, there exists a path containing e of the same length as P that starts at v, and ends at w.

Definition 3.2. Let ξ be a positive constant and let G_1 be a graph with property $\mathcal{RE}(\xi)$. We say that a graph G_2 *complements* G_1, if for every path P with a fixed edge e, (i) there exists a path containing e longer than P in the graph $G_1 \cup P$, or (ii) there exist $v \in S_P$ and $w \in T_v$, such that $\{v, w\}$ is an edge of $G_1 \cup G_2 \cup P$ (the sets S_P and T_v are as defined in Definition 3.1).

The next proposition asserts that two graphs as in the above two definitions together give Hamiltonicity. In fact, we will obtain a slightly stronger property which is called *Hamilton connectivity*. A graph is said to be *Hamilton connected* if for every pair of vertices x and y, there exists a path of length $n-1$ that has x and y as its two endpoints. Since a Hamilton connected graph is necessarily non-empty, by taking x and y as two endpoints of an edge in the graph, we can see that Hamilton connectivity implies Hamiltonicity.

Proposition 3.3. *Let ξ be a positive constant. If $G_1 \in \mathcal{RE}(\xi)$ and G_2 complements G_1, then $G_1 \cup G_2$ is Hamilton connected.*

Proof. Let v_1 and v_2 be two arbitrary vertices. If $\{v_1, v_2\}$ is not an edge of $G_1 \cup G_2$, then let G_2' be the graph obtained by adding the edge $e = \{v_1, v_2\}$ to the graph G_2, otherwise let $G_2' = G_2$. Note that G_2' complements G_1. Let P be a longest path in $G_1 \cup G_2'$ which contains e (say it has length ℓ). By Definition 3.2, there exist vertices $v' \in S_P$ and $w' \in T_{v'}$ such that $\{v', w'\}$ is an edge of $G_1 \cup G_2'$ (where the sets S_P and T_v are as in Definition 3.1). Thus we can find a cycle containing e, of length ℓ.

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If this cycle is not a Hamilton cycle, then by the connectivity of G_1, there exists a vertex x not in the cycle, which is incident to some vertex of the cycle. There are two ways to construct a path of length $\ell + 1$ using this edge and the cycle, and one of them necessarily contains the edge e. Since this contradicts the maximality of P, the cycle must have been a Hamilton cycle. By removing the edge e, we get a Hamilton path connecting v_1 and v_2 in $G_1 \cup G_2$. □

Thus our strategy for proving Hamiltonicity is to find a subgraph with property $\mathcal{RE}(\xi)$ and a one that complements it. In the remainder of the subsection, we provide a list of deterministic properties, which when satisfied, imply property $\mathcal{RE}(\xi)$. After establishing this lemma, later it will suffice to verify that these deterministic properties hold for the graphs we are interested in.

Definition 3.4. Let ε and r be positive reals. A graph G is a *half-expander with parameters ε and r*, if the following properties hold.

(i) For every set X of vertices of size $|X| \leq \frac{\varepsilon n}{r}$, $|N(X)| \geq r|X|$,
(ii) for every set X of vertices of size $|X| \geq \frac{n}{\varepsilon r}$, $|N(X)| \geq (\frac{1}{2} - \varepsilon)n$, and
(iii) for every pair of disjoint sets X, Y such that $|X|, |Y| \geq (\frac{1}{2} - \varepsilon^{1/5})n$, $e(X, Y) > 2n$.

Lemma 3.5. *There exists a positive ε_0 such that for every positive $\varepsilon \leq \varepsilon_0$, the following holds for every $r \geq 16\varepsilon^{-3} \log n$: every half-expander with parameters ε and r has property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$.*

Proof. For simplicity of notation, we assume that we are given a half-expander with parameters ε^4 and r, and will prove that it has property $\mathcal{RE}(\frac{1}{2} + \varepsilon^4)$. Let $\varepsilon_0 = 25^{-5}$, and suppose that we are given positive reals $\varepsilon \leq \varepsilon_0$ and $r \geq 16\varepsilon^{-12} \log n$. Let G be a half-expander with parameters ε^4 and r. To prove that G is connected, take two vertices v and w. By Properties (i) and (ii) of Definition 3.4, there exist sets A_v and A_w each of size at least $(\frac{1}{2} - \varepsilon^4)n \geq (\frac{1}{2} - \varepsilon^{4/5})n$ such that the connected component containing v contains A_v, and the connected component containing w contains A_w. There exists an edge between A_v and A_w by Property (iii) of Definition 3.4. Consequently, there exists a path between v and w in G for all pairs of vertices v and w.

Let $P = (v_0, \cdots, v_\ell)$ be a path with some fixed edge $e_f = \{v_f, v_{f+1}\}$, and let F be the set $\{v_{f-1}, v_f, v_{f+1}, v_{f+2}\}$. If there is a path longer than P that contains e_f in the graph $G \cup P$, then there is nothing to prove since it satisfies the first condition of Definition 3.1. Thus we may assume that P is a longest path in $G \cup P$ that contains e_f. We start by rotating v_0 to construct the set S_P as in Definition 3.1. Afterwards, we will construct the sets T_v the same way.

For a subset $X = \{v_{a_1}, v_{a_2} \cdots, v_{a_i}\}$ of vertices of P, let $X^- = \{v_{a_1-1}, v_{a_2-1}, \cdots, v_{a_i-1}\}$ and $X^+ = \{v_{a_1+1}, v_{a_2+1}, \cdots, v_{a_i+1}\}$ (if the index becomes either -1 or $\ell + 1$, then remove the corresponding vertex from the set X^- or X^+). Throughout the proof we will repeatedly consider the operation X^+ and X^- for various sets X. While performing this operation, one must take special care of the vertices which lie in the boundary of the intervals P_i. However, we will ignore the effect of the boundary vertices, since it will simplify the computation, and will only affect it by some small order terms. Let $k = 4\varepsilon^{-4} \log n$, and partition the path P into k consecutive intervals of lengths as equal as possible. Denote these intervals as $P_1, \cdots, P_k$.

Step 1: *Initial rotations.*

$$v \quad P_2 \quad P_6 \quad P_5 \quad P_4 \quad P_{11} \quad P_{12} \quad P_8 \quad P_{15} \quad P_{14} \quad v_\ell$$
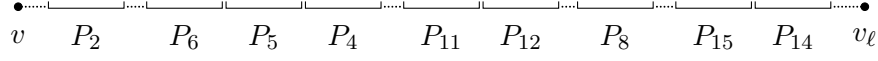
Figure 3.2: A path obtained by several rounds of rotations. The order of the intervals have been changed as a consequence of these rotations. Intervals that do not appear in the figure ($P_1, P_3, P_7, P_9, P_{10}, P_{13}, P_{16}, \cdots$) contain broken edges, and the vertices in those intervals will be spread out among the dotted area. Note that non-broken intervals appear either in the original order, or in the reversed order. By universality, even if we change the vertex v into another vertex, the order of the non-broken intervals $P_2, P_6, P_5, \cdots$ as above will not change.

Our argument is based on that of Sudakov and Vu [32] where one performs rotations and extensions in a very controlled manner. Let $S_0 = \{v_0\}$. We will iteratively construct sets S_i for $i \geq 0$ so that $|S_i| = \varepsilon^{-8i}$, and for all $v \in S_i$, there exists a path of length ℓ which starts at v, ends at v_ℓ, contains e_f, and has been obtained from P by i rounds of rotations. We will continue to construct sets as long as $|S_i| \leq \frac{\varepsilon^4 n}{r}$. Note that this implies $i \leq \log n$. For a vertex $v \in S_i$, let $e_{v,1}, \cdots, e_{v,i}$ be the broken edges created in constructing the path from v to v_ℓ, in the order they were broken (we call them the *broken edges* of v). Note that the order in which the broken edges were created is not necessarily the same as the order in which they appear along the path. We impose the following conditions on S_i:

(Universality) for all $1 \leq a \leq i$, there exists an index j_a such that for all $v \in S_i$, the edge $e_{v,a}$ belongs to P_{j_a}. Moreover, if several broken edges belong to the same interval, then the order in which they appear within the interval also does not depend on v.

An important byproduct of this property is that for every interval P_j which does not contain a broken edge, there is a fixed orientation so that for all $v \in S_i$, the path from v to v_ℓ traverses P_j in this orientation. Moreover, the order in which each non-broken interval appears along these paths does not depend on v (thus is *universal*, see Figure 3.2).

Assume we have completed constructing the set S_i which has the properties listed above. Let S_{i+1}^0 be the set $N(S_i) \setminus (F \cup \bigcup_{a=1}^i (S_a \cup S_a^- \cup S_a^+))$. If there is a vertex in S_{i+1}^0 which is not in the path P, then we can use it to find a path longer than P that contains the edge e_f. Therefore we may assume that $S_{i+1}^0 \subset V(P)$. Since we removed all the vertices belonging to S_a, S_a^-, S_a^+ for $a \leq i$ when defining S_{i+1}^0, all the vertices in S_{i+1}^0 can be used as pivot points to create new endpoints (note that the broken edges obtained by this procedure are necessarily distinct from all the previous broken edges). Since $|N(S_i)| \geq r|S_i|$, we have the following estimate on the size of S_{i+1}^0:

$$|S_{i+1}^0| \geq r|S_i| - 3\sum_{a=1}^i |S_a| - 4 \geq r\varepsilon^{-8i} - 3\varepsilon^{-8(i+1)} - 4 \geq \left(\frac{r}{2}\right)\varepsilon^{-8i}. \tag{3.1}$$

It now suffices to choose a suitable subset of S_{i+1}^0 which also satisfies universality.

Pick an arbitrary $v \in S_i$, and for each P_j, let $w(j)$ be the number of broken edges of v that P_j

contains (note that by the universality, the choice of v does not matter). Note that

$$\sum_{j=1}^{k}(w(j)+1) = i + k \leq \left(\frac{4}{\varepsilon^4}+1\right)\log n \leq \frac{\varepsilon^8 r}{2}.$$

Consequently, there is an index $j_* \in [k]$ such that P_{j_*} contains at least $\frac{2(w(j_*)+1)}{\varepsilon^8 r}$ proportion of the vertices of the set S_{i+1}^0. In other words we have

$$|P_{j_*} \cap S_{i+1}^0| \geq \left(\frac{r}{2}\right)\varepsilon^{-8i} \cdot \frac{2(w(j_*)+1)}{\varepsilon^8 r} = \varepsilon^{-8(i+1)} \cdot (w(j_*)+1)).$$

By using the vertices in $P_{j_*} \cap S_{i+1}^0$ as pivot points, we can obtain a set of new endpoints S_{i+1}' with $|S_{i+1}'| \geq \varepsilon^{-8(i+1)} \cdot (w(j_*)+1)$. By construction, all the newly added pivot points and corresponding broken edges belong to the same interval P_{j_*}. Therefore, it suffices to find a large subset S_{i+1} of S_{i+1}' such that the broken edges of these vertices that belong to P_{j_*} appear in some universal order (note that this automatically is true for indices other than j_* by the same property for S_i).

By definition, for $h = w(j_*)$, there exist indices $i_1, i_2, \cdots, i_h$ of value at most i such that for every $v \in S_{i+1}'$, the broken edges $e_{v,i_1}, \cdots, e_{v,i_h}, e_{v,i+1}$ are in P_{j_*} ($e_{v,i+1}$ is the newly created broken edge in the $(i+1)$-th round). By the hypothesis, we know that $e_{v,i_1}, \cdots, e_{v,i_h}$ appear in some fixed order which does not depend on v. There are $h+1$ relative positions that $e_{v,i+1}$ can lie within that ordering. We let S_{i+1} be a subset of S_{i+1}' of size at least $\frac{|S_{i+1}'|}{h+1} = \frac{|S_{i+1}'|}{w(j_*)+1}$ such that for all the vertices in this set, the new broken edge has the same relative order in P_{j_*} with respect to the edges $e_{v,i_1}, \cdots, e_{v,i_h}$. This choice of S_{i+1} satisfies all our assumptions, and we have $|S_{i+1}| \geq \varepsilon^{-8(i+1)}$. Redefine S_{i+1} as an arbitrary subset of size exactly $\varepsilon^{-8(i+1)}$. Repeat the above until we have a set S_{t-1} of size at least $\frac{\varepsilon^4 n}{r}$ and redefine S_{t-1} as an arbitrary subset of size exactly $\frac{\varepsilon^4 n}{r}$. Repeat the above process one more time to obtain a set S_t of size exactly $\frac{n}{\varepsilon^4 r}$ (note that $t \leq \log n$).

Step 2: *Terminal rotations.*

Let $k' = k - t - 2$. There are at most $t + 2$ intervals which contain at least one broken edge for some vertex of S_t, or intersects F, and thus at least k' intervals do not have this property. For notational convenience, relabel the intervals so that the intervals $P_1, \cdots, P_{k'}$ contain no broken edges and do not intersect F, and let $P' = P_1 \cup \cdots \cup P_{k'}$. Note that

$$|V(P) \setminus P'| \leq (t+2)\left(\frac{n}{k}+1\right) \leq (\log n + 2)\left(\frac{n}{k}+1\right) \leq \frac{4n\log n}{k} \leq \varepsilon^4 n.$$

Further assume that for $1 \leq i \leq k'$, each path from $v \in S_t$ to v_ℓ traverses the interval P_i positively (we lose some generality here, but we use no properties of this special case, and the assumption is made just for the sake of clarity of presentation). Define S_P as the collection of vertices $v \in P$ which have the property that in $G \cup P$ there exists a path of length ℓ containing e_f that starts at v and ends at v_ℓ. Note that S_P contains S_t.

We want to show that $|S_P| \geq (\frac{1}{2}+\varepsilon^4)n$. Assume to the contrary that $|S_P| < (\frac{1}{2}+\varepsilon^4)n$. We claim that under this assumption, the inequality $|S_P^+ \Delta S_P^-| \leq 22\varepsilon n$ holds. The proof of this claim will be given later (see Claim 3.6). Given this claim, consider the set $Z = P' \setminus (S_P^+ \cup S_P^-)$. We have $|S_P^+ \cup S_P^-| \leq |S_P| + 22\varepsilon n \leq (\frac{1}{2}+23\varepsilon)n$. Since $|V(P) \setminus P'| \leq \varepsilon^4 n$, this implies $|Z| \geq$

$|V(P)| - (\frac{1}{2} + 23\varepsilon)n - \varepsilon^4 n \geq |V(P)| - (\frac{1}{2} + 24\varepsilon)n$, and for the set $Z' := Z \cap (P')^- \cap (P')^+$, we have $|Z'| \geq |Z| - 2k' \geq |V(P)| - (\frac{1}{2} + 25\varepsilon)n$. Note that if some vertex $v \in S_P$ is adjacent to some vertex $w \in Z'$ and both $\{w^-, w\}$, $\{w, w^+\}$ have not been broken while obtaining v as an endpoint, then we obtain a contradiction since this necessarily gives w^- or w^+ as a new endpoint, which by the definition of Z' is not in S_P.

Let $Y = N(S_t) \cap P'$. Since the path from $v \in S_t$ to v_ℓ traverses the intervals P_i ($1 \leq i \leq k'$) positively, we can use the vertices of Y as pivot points to construct endpoints Y^-. We have $|Y^-| \geq |N(S_t)| - (\varepsilon^4 n + k) \geq (\frac{1}{2} - 3\varepsilon)n$. For each vertex $y \in Y^-$, fix one path of length ℓ which starts at y, ends at v_ℓ, and has the property that all the broken edges but the last one lie outside of P'. Thus at most one broken edge will lie inside P'. Consequently, if some vertex $y \in Y^-$ has at least three neighbors in Z', then we necessarily have a vertex $w \in Z'$ for which both $\{w, w^-\}$ and $\{w, w^+\}$ are not broken edges of y. Then by the observation made in the previous paragraph we reach a contradiction. Furthermore, by the maximality of P, we know that there are no edges between Y^- and $V \setminus V(P)$. Therefore, for the set $Z'' = Z' \cup (V \setminus V(P))$ which is of size $|Z''| \geq (|V(P)| - (\frac{1}{2} + 25\varepsilon)n) + (|V| - |V(P)|) \geq (\frac{1}{2} - 25\varepsilon)n$, there are no edges between Y^- and Z''. However, since $|Y^-|$ and $|Z''|$ are both at least $(\frac{1}{2} - 25\varepsilon)n$, by property (iii) of Definition 3.4, there exist more than $2n$ edges between Y^- and Z'' and therefore some vertex in Y^- must have at least three neighbors in Z''. Consequently, we must have had $|S_P| \geq (\frac{1}{2} + \varepsilon^4)n$.

Step 3: *Rotating the other endpoint.*

For every $v \in S_P$, there exists a path containing e_f of length ℓ which starts at v and ends at v_ℓ. Now by repeating the above for the other endpoint v_ℓ, we can see that for every $v \in S_P$, there exists a set T_v of size at least $(\frac{1}{2} + \varepsilon^4)n$ such that for every $w \in T_v$, there exists a path of length ℓ which starts at v, ends at w, and contains the edge e_f. $\qquad\square$

It remains to prove the claim. The intuition behind this perhaps strangely looking claim comes from the following two non-Hamiltonian graphs whose minimum degrees are slightly less than $\frac{n}{2}$. First, consider the graph consisting of two disjoint cliques of size $\frac{n}{2}$ connected by a single edge, and consider a Hamiltonian path in it. It is not too difficult to see that by rotating the starting point we only get the first half of the path as new starting points. More precisely, using the same notation as in the proof above, we will get $|S_P| = \frac{n}{2} - 1$ and $|S_P^+ \Delta S_P^-| = 3$. Second, consider a complete bipartite graph in which one part A has one more vertex than the other part B, and consider a Hamiltonian path in it (it must be an A-A path). In this case, by rotating the starting point we only get the vertices in A as new starting points, and therefore $|S_P| = \frac{n}{2}$ and $|S_P^+ \Delta S_P^-| = 0$. Note that the two graphs above both have $|S_P|$ close to $\frac{n}{2}$ and $|S_P^+ \Delta S_P^-|$ small, but for very different reasons. Our claim asserts that, in general, if the given graph has $|S_P|$ close to $\frac{n}{2}$, then it indeed is true that the graph has small $|S_P^+ \Delta S_P^-|$.

Claim 3.6. *If $|S_P| < (\frac{1}{2} + \varepsilon^4)n$, then $|S_P^+ \Delta S_P^-| \leq 22\varepsilon n$.*

Proof. Recall that we will ignore the effect of the boundary vertices while performing the operations X^- and X^+ for sets X, since it will simplify the computation, and will only affect it by some small order terms. The main strategy is as following. We first rotate the path P in two ways to obtain some set Q of endpoints in two different ways while keeping a big chunk P'' of P not broken. For

each endpoint w in Q, the two paths will traverse P'' in opposite direction (see Figure 3.3). If this is the case, then both sets $(N(Q) \cap P'')^-$ and $(N(Q) \cap P'')^+$ become subsets of S_P. From this we will conclude that the two sets S_P^+ and S_P^- do not differ too much.

We follow the same notation as in the proof above. Recall that the set S_P was defined as the collection of vertices v in P for which there exists a path of length ℓ starting at v and ending at v_ℓ that contains some fixed edge e_f. Recall that $P' = P_1 \cup \cdots \cup P_{k'}$, $|V(P) \setminus P'| \leq \varepsilon^4 n$, and that by property (ii) of Definition 3.4, for every set X of size $|X| \geq \frac{n}{\varepsilon^4 r}$, we have $|N(X)| \geq (\frac{1}{2} - \varepsilon^4)n$. Note that by using the vertices in $N(S_t) \cap P'$ as pivot points, we get $|S_P \cap P'| \geq |N(S_t) \cap P'| \geq |N(S_t)| - |V(P) \setminus P'| \geq (\frac{1}{2} - 2\varepsilon^4)n$. For a subset I of $[k']$, we define $P_I = \cup_{i \in I} P_i$.

We first make a simple observation. Let X be some set of endpoints obtained by rotating the given path P, where $|X| \geq \frac{n}{\varepsilon^4 r}$. Our rotations have been carefully performed, hence in the next round of rotation, many vertices in $N(X)$ will give rise to vertices in S_P. Thus if $|N(X)|$ is close to $\frac{n}{2}$, then since $|S_P| < \left(\frac{1}{2} + \varepsilon^4\right)n$, we will recover most of the vertices in S_P in the next round of rotation. The following simple proposition formalizes this intuition and will be used several times in proving the claim.

Proposition 3.7. *Let X be a subset of S_P of size $|X| \geq \frac{n}{\varepsilon^4 r}$, and for every $v \in X$, fix one path of length ℓ containing e_f from v to v_ℓ. Let I and J be disjoint subsets of $[k']$, and assume that for every $\eta \notin I$ there exists an orientation o_η such that for every $v \in X$, there exists a path from v to v_ℓ that has no broken edge in P_η, and traverses P_η in direction o_η. Then*

$$|N(X) \cap P_J| \geq |S_P \cap P_J| - |P_I| - 3\varepsilon^4 n.$$

Proof. For every $\eta \notin I$, since all the paths traverse P_η in direction o_η, we know that when a vertex in $N(X) \cap P_\eta$ is used as a pivot point, it will create a broken edge in a fixed direction (to the left of the pivot point if o_η is positive, and to the right of the pivot point otherwise). Therefore we have

$$|N(X) \cap P_{[k'] \setminus (I \cup J)}| + |S_P \cap P_I| + |S_P \cap P_J| \leq |S_P|.$$

Consequently,

$$\begin{aligned}
|N(X)| &= |N(X) \cap \overline{P'}| + |N(X) \cap P_I| + |N(X) \cap P_J| + |N(X) \cap P_{[k'] \setminus (I \cup J)}| \\
&\leq |V(P) \setminus P'| + |P_I| + |N(X) \cap P_J| + (|S_P| - |S_P \cap P_I| - |S_P \cap P_J|),
\end{aligned}$$

and by our assumption that $|S_P| \leq \left(\frac{1}{2} + \varepsilon^4\right)n$, we have

$$|N(X)| \leq \left(\frac{1}{2} + 2\varepsilon^4\right) n + |P_I| + |N(X) \cap P_J| - |S_P \cap P_J|.$$

On the other hand, since $|X| \geq \frac{n}{\varepsilon^4 r}$, we have $|N(X)| \geq (\frac{1}{2} - \varepsilon^4)n$. By combining the two bounds we get $|S_P \cap P_J| - |P_I| - 3\varepsilon^4 n \leq |N(X) \cap P_J|$. $\qquad \square$

Let a_1 be the smallest positive integer such that for $k_1 = k' - \varepsilon k \cdot a_1$, there exist at least $2\varepsilon^2 n$ elements of S_P in $Q_1 = P_{[k_1 + 1, k_1 + \varepsilon k]}$ (note that $|Q_1| \leq \varepsilon n$). By construction, there exist at most $2\varepsilon n$ elements of S_P in $P_{[k_1 + \varepsilon k + 1, k']}$, and at least $|S_P \cap P'| - 2\varepsilon n - |Q_1| \geq (\frac{1}{2} - 5\varepsilon)n$ vertices in $P_{[1, k_1]}$. Let
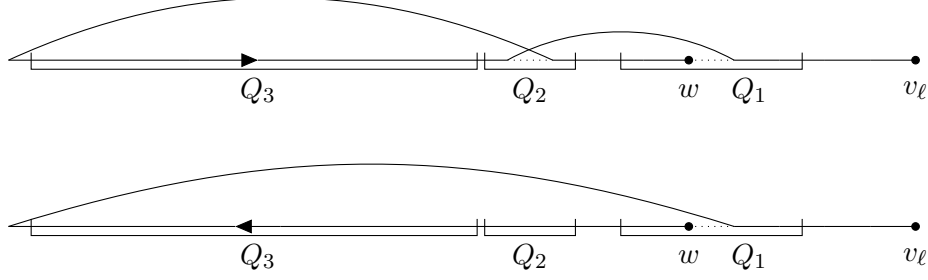
Figure 3.3: Rotating in two different ways to get the same endpoint.

a_2 be the smallest positive integer such that for $k_2 = k_1 - \varepsilon^2 k \cdot a_2$, there exist at least $\varepsilon^3 n$ points of S_P in $Q_2 = P_{[k_2+1,k_2+\varepsilon^2 k]}$ (note that $|Q_2| \le \varepsilon^2 n$). Note that there exist at most εn vertices of S_P in $P_{[k_2+\varepsilon^2 k+1,k_1]}$. Let $Q_3 = P_{[1,k_2]}$.

We defined the sets Q_1 and Q_2 so that the numbers of vertices of S_P in both of these sets are quite large. Our goal now is to find a large number of vertices in $Q_1 \cap S_P$ that can be obtained by two different rotations, one rotation giving a path that traverses Q_3 positively and the other giving a path that traverses Q_3 negatively. To do this, first, we will perform two rotations, where we begin by finding endpoints in Q_2 using S_t, and then use these endpoints to find endpoints in Q_1. Second, we will directly rotate from S_t to obtain endpoints in Q_1. Since both ways will give a big proportion of vertices in $Q_1 \cap S_P$, we eventually will find the set of vertices that we wanted.

We will use Proposition 3.7 to formalize this idea. Recall that for $1 \le i \le k$, each path from $v \in S_t$ to v_ℓ traverses the interval P_i positively. Let $Y_2 = N(S_t)^- \cap Q_2$. By our construction we have that for $J = [k_2 + 1, k_2 + \varepsilon^2 k]$, $|S_P \cap P_J| \ge \varepsilon^3 n$. Thus, by Proposition 3.7 with $X = S_t$, $I = \emptyset$, and $J = [k_2 + 1, k_2 + \varepsilon^2 k]$, we see that $|Y_2| = |N(S_t) \cap Q_2| \ge (\varepsilon^3 - 3\varepsilon^4)n$. Since we can use the points in $N(S_t) \cap Q_2$ as pivot points to get new endpoints Y_2, we have that for every $v \in Y_2$, there exists a path of length ℓ containing e_f which starts at v and ends at v_ℓ. Moreover, these paths have exactly one broken edge inside P', and it is in Q_2. Thus we can apply Proposition 3.7 again with $X = Y_2$, $I = [k_2 + 1, k_2 + \varepsilon^2 k]$, and $J = [k_1 + 1, k_1 + \varepsilon k]$ to get $|N(Y_2)^- \cap Q_1| \ge |S_P \cap Q_1| - \varepsilon^2 n - 3\varepsilon^4 n$. For every vertex $v \in N(Y_2)^- \cap Q_1$, there exists a path of length ℓ containing e_f which starts at v and ends at v_ℓ. These paths have exactly two broken edges inside P', both in $Q_1 \cup Q_2$. Moreover, for every interval in Q_3, all these paths traverse the interval in the positive direction. Now apply Proposition 3.7 with $X = S_t$, $I = \emptyset$, and $J = [k_1 + 1, k_1 + \varepsilon k]$ to get $|N(S_t)^- \cap Q_1| \ge |S_P \cap Q_1| - 3\varepsilon^4 n$. The vertices in $N(S_t)^- \cap Q_1$ have similar properties to those in $N(Y_2)^- \cap Q_1$, but the paths for the vertices $N(S_t)^- \cap Q_1$ traverse the intervals in Q_3 in the negative direction (see Figure 3.3).

Since both $N(Y_2)^- \cap Q_1$ and $N(S_t)^- \cap Q_1$ are subsets of S_P and since we constructed Q_1 such that $|S_P \cap Q_1| \ge 2\varepsilon^2 n$, we have for $Y_1 = N(S_t)^- \cap N(Y_2)^- \cap Q_1$,

$$|Y_1| \ge |S_P \cap Q_1| - \varepsilon^2 n - 6\varepsilon^4 n \ge (\varepsilon^2 - 6\varepsilon^4)n.$$

Note that by using Proposition 3.7 with $X = Y_1$, $I = [k_1 + 1, k_1 + \varepsilon k] \cup [k_2 + 1, k_2 + \varepsilon^2 k]$, $J = [1, k_2]$, we get

$$|N(Y_1) \cap Q_3| \ge |S_P \cap Q_3| - |Q_1| - |Q_2| - 3\varepsilon^4 n \ge |S_P \cap Q_3| - 5\varepsilon n.$$

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By the observations made in the previous paragraph, we see that for every $v \in Y_1$, there exist two paths of length ℓ starting at v and ending at v_ℓ such that both paths have at most two broken edges in P', all belonging to $Q_1 \cup Q_2$. Furthermore, one of the paths traverses the intervals in Q_3 positively, and the other negatively. Therefore both $N(Y_1)^- \cap Q_3$ and $N(Y_1)^+ \cap Q_3$ are subsets of S_P.

The above implies that $|(N(Y_1)^- \cap Q_3)\Delta(S_P \cap Q_3)| \leq 5\varepsilon n$. Thus $|(N(Y_1) \cap Q_3)\Delta(S_P^+ \cap Q_3)| \leq 5\varepsilon n$. Similarly we have $|(N(Y_1) \cap Q_3)\Delta(S_P^- \cap Q_3)| \leq 5\varepsilon n$. By the triangle inequality we get

$$|(S_P^- \cap Q_3)\Delta(S_P^+ \cap Q_3)| \leq 10\varepsilon n.$$

Since, by our construction, there are at most $2\varepsilon n$ vertices of S_P in $P_{[k_1+\varepsilon k+1,k']}$ and at most εn vertices of S_P in $P_{[k_2+\varepsilon^2 k+1,k_1]}$, we can conclude that $|S_P \cap \overline{Q_3}| \leq |V(P) \setminus P'| + |Q_1| + |Q_2| + 3\varepsilon n \leq 6\varepsilon n$. This implies that

$$|S_P^- \Delta S_P^+| \leq |(S_P^- \cap Q_3)\Delta(S_P^+ \cap Q_3)| + 2|S_P \cap \overline{Q_3}| \leq 22\varepsilon n,$$

and completes the proof. $\qquad\square$

3.2 Rotation-extension for bipartite graphs

In this subsection, we develop a framework useful in tackling the third case of Lemma 2.1. Note that the given graph in this case has a partition of its vertex set so that there are only few non-adjacent pairs between the two parts. The following definition gives a nice structure that can be used in this kind of graphs.

Definition 3.8. Let G be a graph over a vertex set V, and let $V_1 \cup V_2$ be a partition of V satisfying $|V_1| = |V_2| + k$ for some non-negative integer k.

(i) A tuple (V_1, V_2, S_V, S_E) is a *special frame* of G if S_V is a subset of V_1 of size k, and S_E is a set of k vertex-disjoint edges of G in V_1 such that each vertex in S_V is incident to exactly one edge in S_E. We refer to S_V as the set of *special vertices*, and S_E is the set of *special edges*. Let the *framed subgraph* of G be the subgraph induced by the edges between (V_1, V_2). Let $V_1' = V_1 \setminus S_V$, and $V_1'' = V_1 \setminus V(S_E)$.

(ii) A tuple (V_1, V_2, S_V, S_E, f) is a *matched special frame* of G if (V_1, V_2, S_V, S_E) is a special frame of G, and f is a perfect matching between the vertices of V_1' and V_2. For $v \in V_1' \cup V_2$, we let $f(v)$ be the vertex matched to v in this matching.

Throughout this subsection, we fix a vertex set V on n vertices and a partition $V_1 \cup V_2$ of it satisfying $|V_1| = |V_2| + k$ for some non-negative integer k. We then assume that all the graphs appearing in this subsection are defined over V (therefore if there are several graphs under consideration, then they share the same vertex set and its partition). Suppose that we are given a graph with some special frame (V_1, V_2, S_V, S_E). If $k = 0$, then it suffices to use the edges between the two parts to find a Hamilton cycle. However, if $k > 0$, then we must use some edges within V_1, and the special edges will be these edges.

To construct a Hamilton cycle in graphs with a matched special frame, it is easier to consider only a certain class of paths and cycles.

Definition 3.9. Let (V_1, V_2, S_V, S_E, f) be a matched special frame of some graph. We say that a path P is a *proper path with respect to the frame* if it satisfies the following properties: (i) $V_1' \cap V(P) = f(V_2 \cap V(P))$, (ii) if P contains a special vertex, then it also contains the special edge incident to it, and (iii) P consists only of edges that intersect both V_1 and V_2 and of special edges. We also say that a cycle C is a *proper cycle with respect to the frame* if it satisfies properties (i),(ii), and (iii) above (with C replacing P). We simply say that a path or cycle is *proper* if the frame is clear from the context.

Note that a proper path always has one of its endpoints in V_1, and the other in V_2. Indeed, suppose that P is a proper path with s special edges. Then by properties (i) and (iii), P has length $2|V_2 \cap V(P)| + s - 1$, and thus switches between V_1 and V_2 in total $2|V_2 \cap V(P)| - 1$ times. Since this is an odd number, we can see that the above holds.

We now specify the roles of the graphs performing rotations and extensions.

Definition 3.10. Let ξ be a positive constant. A graph G has property $\mathcal{RE}_b(\xi)$ if it contains a matched special frame whose framed subgraph is connected, and for every proper path P, at least one of the following holds: (i) there exists a proper path longer than P in $G \cup P$, or (ii) there exists a set of vertices $S_P \subset V_2$ of size at least $|S_P| \geq \xi n$ such that for every vertex $v \in S_P$, there exists a set $T_v \subset V_1$ of size $|T_v| \geq \xi n$ such that for every $w \in T_v$, there exists a proper path of the same length as P in $G \cup P$ that starts at v and ends at w.

Definition 3.11. Let ξ be a positive constant, and let G_1 be a graph with property $\mathcal{RE}_b(\xi)$. We say that a graph G_2 *complements* G_1, if for every proper path P, (i) there exists a proper path longer than P in $G_1 \cup P$, or (ii) there exist vertices $v \in S_P$ and $w \in T_v$ such that $\{v, w\}$ is an edge of $G_1 \cup G_2 \cup P$ (the sets S_P and T_v are as defined in Definition 3.10).

Two graphs as in the above two definitions together give Hamiltonicity.

Proposition 3.12. *Let ξ be a positive constant. If $G_1 \in \mathcal{RE}_b(\xi)$, and G_2 complements G_1, then $G_1 \cup G_2$ is Hamiltonian.*

Proof. Let f be the matching in the matched special frame of G_1 as in Definition 3.8, and P be a longest proper path in $G_1 \cup G_2$. Note that we can use the frame of G_1 also as a frame of $G_1 \cup G_2$. Since G_2 complements G_1, by Definition 3.11, we can find a proper cycle $(v_0, \cdots, v_\ell, v_0)$ in $G_1 \cup G_2$ over the vertex set of P. Assume that this cycle is not Hamiltonian. Then by the connectivity of the framed subgraph of G_1, there exists an edge of the form $\{x, v_i\}$ for some vertex x not in the cycle, where x and v_i belong to different parts of the frame. We claim that this violates the maximality assumption on P. This will imply that the graph $G_1 \cup G_2$ is Hamiltonian.

Indeed, either the vertex x is a special vertex or not. If x is a special vertex, then since $\{x, v_i\}$ cannot be a special edge, we must have a special edge $\{x, x'\}$ for some $x' \neq v_i$. If x' is on the path P, then we immediately obtain a proper path longer than P. Thus we may assume that x' is not on the path. In this case we can use the path $(v_i, x, x', f(x'))$ to find a longer proper path (note that $f(x')$ is also not in the path P by the definition of a proper path). Finally, if x is not a special vertex, then we can use the path $(v_i, x, f(x))$ to find a longer proper path (again, $f(x)$ is not in the path P by the definition of a proper path). Thus in any case, we deduce a contradiction. $\square$

As in the previous subsection, we provide a list of deterministic properties which when satisfied, imply property $\mathcal{RE}_b(\xi)$.

Definition 3.13. Let ε, r be positive constants, and let k be a non-negative integer. We say that a graph G is a *k-bipartite-expander with parameters ε and r*, if it contains a special frame (V_1, V_2, S_V, S_E) with $|V_1| = |V_2| + k$ such that the following properties hold:

(i) For all $X \subset V_1$, if $|X| \leq \frac{n}{r^{3/2}}$, then $|N(X) \cap V_2| \geq r|X|$, and if $|X| \geq \frac{n}{r^{3/4}}$, then $|N(X) \cap V_2| \geq (1 - \varepsilon)|V_2|$, and

(ii) for all $Y \subset V_2$, if $|Y| \leq \frac{n}{r^{3/2}}$, then $|N(Y) \cap V_1''| \geq r|Y|$, and if $|Y| \geq \frac{n}{r^{3/4}}$, then $|N(Y) \cap V_1''| \geq (1 - \varepsilon)|V_1''|$.

We often refer to k-bipartite-expanders as *bipartite-expanders* when k is clear from the context.

Lemma 3.14. *For every positive reals $r \geq 16$, $\varepsilon \leq \frac{1}{4}$, and non-negative integer $k \leq \frac{n}{30}$, every k-bipartite-expander with parameters ε and r has property $\mathcal{RE}_b(\frac{1}{6})$.*

Proof. Let G be the given bipartite-expander with parameters ε and r, and let (V_1, V_2, S_V, S_E) be its special frame. Throughout the proof, we consider only the edges that belong to the framed subgraph and the special edges. Given properties (i) and (ii) of Definition 3.13, it follows that every connected component of the framed subgraph of G contains at least $\frac{3}{4}|V_2|$ vertices of V_2. Consequently, every two connected components intersect, and the framed subgraph of G is connected. We then verify Hall's condition to find a matching f between the vertices of V_1' and V_2. By property (i) of bipartite-expanders, we have $|N(X) \cap V_2| \geq |X|$ for every subset X of V_1 of size at most $\frac{3}{4}|V_2|$. Then by property (ii), we have $|N(Y) \cap V_1'| \geq |N(Y) \cap V_1''| \geq |Y|$ for every subset Y of V_2 of size at most $\frac{3}{4}|V_1''|$. Now suppose that Hall's condition is not satisfied, and that there is some subset X of V_1' of size larger than $\frac{3}{4}|V_2|$ that satisfies $|X| > |N(X)| \geq \frac{3}{4}|V_2|$ (the second inequality follows from the fact we established above for subsets of size at most $\frac{3}{4}|V_2|$). Then it implies that all the neighbors of $V_2 \setminus N(X)$ in V_1' belong to $V_1' \setminus X$. However, since

$$|V_2 \setminus N(X)| = |V_2| - |N(X)| \leq \frac{1}{4}|V_2| < \frac{3}{4}|V_1''|,$$

(we used the fact $k \leq \frac{n}{30}$ in the last inequality) we thus must have $|V_2 \setminus N(X)| \leq |N(V_2 \setminus N(X))| \leq |V_1' \setminus X|$ which is a contradiction since $|V_1' \setminus X| = |V_1'| - |X|$ and $|V_2 \setminus N(X)| = |V_2| - |N(X)| > |V_2| - |X|$. Thus Hall's condition holds, and we may let f be one fixed perfect matching between the vertices of V_1' and V_2. Consider the matched special frame (V_1, V_2, S_V, S_E, f).

Let us now focus on verifying the remaining condition given in Definition 3.10. Let $P = (v_0, \cdots, v_\ell)$ be a proper path of length ℓ where $v_0 \in V_2$ (recall that one endpoint of every proper path is in V_2). For a subset X of vertices of P, we use notations X^- and X^+ as in the proof of Lemma 3.5. We will first construct the set S_P. This will be done by iteratively constructing sets $S_i \subset V_2$ for $i \geq 0$ where S_i has the property that for every $v \in S_i$, there exists a proper path of length ℓ which starts at v and ends at v_ℓ. Moreover, $S_i \subset S_{i+1}$ will hold for every $i \geq 0$.

Let $S_0 = \{v_0\}$. Assume that we have completed constructing S_i for some $i \geq 0$. We first claim that $N(S_i) \cap V_1 \subset V(P)$. If the claim does not hold, then we have an edge $\{x, y\}$ for $x \in S_i$, $y \in V_1 \setminus V(P)$. If y is not a special vertex, then we can find a proper path longer than P by attaching the path

$(f(y), y, x)$ to the proper path of length ℓ starting at x and ending at v_ℓ (note that $f(y)$ is not in P by the definition of a proper path). On the other hand, if y is a special vertex, then let $\{y, y'\}$ be the special edge incident to y. If $y' \in V(P)$, then the edges $\{x, y\}$ and $\{y, y'\}$, together with P immediately give a proper path longer than P. Finally, if $y' \notin V(P)$, then we can use the path $(x, y, y', f(y'))$ to find a longer proper path than P ($f(y')$ is not in P by the definition of a proper path). Thus we indeed must have $N(S_i) \cap V_1 \subset V(P)$.

Now, if a vertex $w \in N(S_i) \cap V_1''$ is not in S_i^- or S_i^+, then w can be used as a pivot point to give either w^- or w^+ as a new endpoint of some proper path that has v_ℓ as its other endpoint (recall that S_i contains the sets S_j for $j \leq i$). Note that it is crucial to consider the set V_1'' as otherwise we might end up breaking some special edges of the path P and the resulting path will no longer be proper. Let S_{i+1} be the union of the set of endpoints obtained in this way and the set S_i. Since an endpoint can be obtained in at most two ways, we have

$$|S_{i+1}| \geq \frac{1}{2}\Big(|N(S_i) \cap V_1''| - |S_i^+| - |S_i^-|\Big) + |S_i| \geq \frac{1}{2}|N(S_i) \cap V_1''|.$$

If $|S_i| \leq \frac{n}{r^{3/2}}$, then we have $|N(S_i) \cap V_1''| \geq r|S_i|$ and therefore, $|S_{i+1}| \geq \frac{r}{2} \cdot |S_i|$. Since $r \geq 16$, at some point t, S_t will have size $|S_t| \geq \frac{n}{r^{3/2}}$. Redefine S_t as an arbitrary subset of it of size $\frac{n}{r^{3/2}}$ and repeat the above once more to get $|S_{t+1}| \geq \frac{r}{2}|S_t| \geq \frac{n}{r^{3/4}}$ (recall that $r \geq 16$). Again redefine S_{t+1} as an arbitrary subset of it of size $\frac{n}{r^{3/4}}$, and repeat the above for the final time, to get a set $S_P = S_{t+2}$ of size

$$|S_P| \geq \frac{1}{2} \cdot (1 - \varepsilon)|V_1''| \geq \frac{3}{8}\left(\frac{n}{2} - \frac{3k}{2}\right) \geq \frac{n}{6},$$

where we used the fact that $\varepsilon \leq \frac{1}{4}$ and $k \leq \frac{n}{30}$. For each vertex $v \in S_P$, we can perform the same process as above to the other endpoint v_ℓ to find a set T_v of size at least $\frac{n}{6}$ satisfying the property of Definition 3.10. $\qquad \square$

In fact, similar arguments also apply to non-bipartite graphs.

Definition 3.15. For positive constants ε and r, we say that a graph G on n vertices is an *expander with parameters ε and r*, if the following properties hold:

(i) For every subset of vertices X of size $|X| \leq \frac{n}{r^{3/2}}$, we have $|N(X)| \geq r|X|$, and

(ii) for every subset of vertices X of size $|X| \geq \frac{n}{r^{3/4}}$, we have $|N(X)| \geq (1 - \varepsilon)n$.

The proof of Theorem 3.14 given above can be easily modified to give the following result.

Lemma 3.16. *For every positive reals $r \geq 16$ and $\varepsilon \leq \frac{1}{4}$, every expander with parameters ε and r has property $\mathcal{RE}(\frac{1}{6})$.*

4   Random subgraphs of Dirac graphs

In this section we prove Theorem 1.1. The following well-known concentration result (see, for example [1], Appendix A) will be used several times throughout the proof. We denote by $Bi(n, p)$ a binomial random variable with parameters n and p.

Theorem 4.1 (Chernoff's Inequality). *If $X \sim Bi(n, p)$ and $\lambda \leq np$, then*

$$P\big(|X - np| \geq \lambda\big) \leq e^{-\Omega(\lambda^2/(np))}.$$

We begin by applying Lemma 2.1 with $\alpha = \frac{1}{2^{23}}$ and $\gamma = \frac{1}{32}$ to classify Dirac graphs into three types. We will show case by case that a random subgraph of a Dirac graph of each type is a.a.s. Hamiltonian.

4.1 First case

Let G be a Dirac graph satisfying $e(A, B) \geq \alpha n^2$ for all half-sets A and B.

Lemma 4.2. *There exist positive reals ε and β such that for every $p \geq \frac{\beta \log n}{n}$ the graph G_p a.a.s. contains a subgraph with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ that has at most $\beta n \log n$ edges.*

Proof. Let $\varepsilon = \min\{\varepsilon_{3.5}, (\frac{\alpha}{4})^5, \frac{1}{2^4 e^2}\}$, and $\beta \geq 512\varepsilon^{-3}$ be a large constant. Let $p' = \frac{\beta \log n}{n}$ and suppose that we are given $p \geq p'$. Let H be a random subgraph of G_p obtained by taking each edge independently with probability $\frac{p'}{p}$, and note that the distribution of H is identical to that of $G_{p'}$. Since $G_{p'}$ a.a.s. has at most $(1 + o(1))e(G)p' \leq \beta n \log n$ edges, it suffices to show that $G_{p'}$ a.a.s. has property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$. By Lemma 3.5, we can prove our claim by verifying that $G_{p'}$ is a.a.s. a half-expander with parameters ε and $r = \frac{\beta}{32} \log n = \frac{np'}{32}$.

We will establish the following four properties of $G_{p'}$ which together will imply that $G_{p'}$ is a.a.s. a half-expander with parameters ε and r (note that $np' = \beta \log n$).

1. $G_{p'}$ a.a.s. has minimum degree at least $\frac{1}{3}np'$.
2. For every pair of sets X and Y satisfying $|X| \leq \frac{\varepsilon n}{r}$ and $|Y| \leq r|X|$, we a.a.s. have $e_{G_{p'}}(X, Y) \leq \frac{1}{4}|X|np'$.
3. For every pair of sets X and Y of size $|X| \geq \frac{n}{\varepsilon r}$ and $|Y| \geq (\frac{1}{2} + \varepsilon)n$, $G_{p'}$ a.a.s. contains at least one edge between X and Y.
4. For every pair of sets X and Y of size $|X|, |Y| \geq (\frac{1}{2} - \varepsilon^{1/5})n$, $G_{p'}$ a.a.s. contains at least $2n + 1$ edges between X and Y.

Indeed, suppose that all four properties hold. Let X be a set of size at most $\frac{\varepsilon n}{r}$ and assume that $|N(X)| \leq r|X|$. For the set $Y = N(X)$, by Properties 1 and 2, we must have

$$\frac{1}{4}|X|np' \geq e_{G_{p'}}(X, Y) = e_{G_{p'}}(X, V(G)) \geq |X| \cdot \frac{1}{3}np'$$

which is a contradiction. Thus Condition (i) of Definition 3.4 holds. Now let Z be a set of size at least $\frac{n}{\varepsilon r}$ and assume that $|N(Z)| \leq (\frac{1}{2} - \varepsilon)n$. If this is the case, then there are no edges between Z and $V \setminus N(Z)$ in $G_{p'}$, and this contradicts Property 3 since $|V \setminus N(Z)| \geq (\frac{1}{2} + \varepsilon)n$. This establishes Condition (ii) of Definition 3.4. Finally, it is easy to see that Property 4 implies Condition (iii) of Definition 3.4.

Now we establish the four properties listed above. Since G has minimum degree at least $\frac{n}{2}$, by Chernoff's inequality, the probability of a fixed vertex having degree less $\frac{1}{3}np'$ is at most $e^{-\Omega(np')} = o(n^{-1})$ for large enough β. By taking the union bound, we obtain Property 1. For Property 2, let

X and Y be two sets of size $|X| = k$ and $|Y| = rk$ (it suffices to prove for Y exactly of size rk). To estimate the probability of $e_{G_{p'}}(X,Y) \geq \frac{knp'}{4}$, we may estimate the probability that at least $\frac{knp'}{8}$ non-ordered pairs of the form $\{v,w\}$ for $v \in X, w \in Y$, become edges of $G_{p'}$. Consequently, the probability of having $e_{G_{p'}}(X,Y) \geq \frac{knp'}{4}$ is at most

$$\binom{rk^2}{knp'/8}(p')^{knp'/8}.$$

Therefore by the union bound, the probability of having two sets violating Property 2 is at most

$$\sum_{k=1}^{\varepsilon n/r} \binom{n}{k}\binom{n}{rk}\binom{rk^2}{knp'/8}(p')^{knp'/8} \leq \sum_{k=1}^{\varepsilon n/r} \binom{n}{k}\left(\frac{en}{rk}\right)^{rk}\left(\frac{8erk}{n}\right)^{knp'/8}$$

$$= \sum_{k=1}^{\varepsilon n/r} \binom{n}{k}\left(\frac{en}{rk}(\frac{8erk}{n})^4\right)^{rk} = \sum_{k=1}^{\varepsilon n/r} \binom{n}{k}\left(\frac{2^{12}e^5 r^3 k^3}{n^3}\right)^{rk}$$

$$\leq \sum_{k=1}^{\varepsilon n/r} \binom{n}{k}\left(\varepsilon^3 2^{12} e^5\right)^{rk}$$

$$\leq \sum_{k=1}^{\varepsilon n/r} \left(\frac{en}{k}\left(\varepsilon^3 2^{12} e^5\right)^r\right)^k,$$

which is $o(1)$ for large enough β since $r = \frac{\beta}{32}\log n = \frac{np'}{32}$, and $\varepsilon \leq 2^{-4}e^{-2}$.

To establish Property 3, let X and Y be two sets as in Property 3. Since the graph G has minimum degree at least $n/2$, the number of edges in G between X and Y is at least

$$\frac{1}{2}|X| \cdot \varepsilon n \geq \frac{n^2}{2r}.$$

Consequently the probability of Property 3 not holding is at most

$$2^{2n} \cdot (1-p')^{n^2/(2r)} \leq 2^{2n} e^{-n^2 p'/(2r)} = 2^{2n} e^{-16n} = o(1)$$

(we used the fact that $p' = \frac{\beta \log n}{n}$ and $r = \frac{\beta}{32}\log n$).

It now remains to verify Property 4. It suffices to consider a pair of sets X and Y which both have size exactly $(\frac{1}{2} - \varepsilon^{1/5})n$. Let X' be an arbitrary set of vertices of size $\frac{n}{2}$ that contains X, and similarly define Y'. Then the graph G satisfies $e_G(X',Y') \geq \alpha n^2$. Therefore we have

$$e_G(X,Y) \geq e_G(X',Y') - |X' \setminus X| \cdot n - |Y' \setminus Y| \cdot n \geq \alpha n^2 - 2\varepsilon^{1/5} n^2 \geq \frac{\alpha n^2}{2},$$

and thus by Chernoff's inequality, the probability that $G_{p'}$ has less than $\frac{\alpha n^2 p'}{4}$ edges between X and Y is at most $e^{-\Omega(\alpha n^2 p')} = e^{-\Omega(\alpha \beta n \log n)}$. Thus by taking the union bound over all possible choices of X and Y, we have Property 4. $\square$

Lemma 4.3. *For every fixed positive reals ε and β, there exists a constant $C = C(\varepsilon, \beta)$ such that the following holds for every $p \geq \frac{C \log n}{n}$: G_p a.a.s. complements every its subgraph with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ that has at most $\beta n \log n$ edges.*

Proof. Let $C \geq \beta$ be a sufficiently large constant. The probability that the assertion of the lemma fails is

$$\mathbf{P} = \mathbf{P}\Big(\bigcup_{R \in \mathcal{RE}(\frac{1}{2}+\varepsilon), |E(R)| \leq \beta n \log n} \Big(\{R \subset G_p\} \wedge \{G_p \text{ does not complement } R\} \Big) \Big)$$

$$\leq \sum_{R \in \mathcal{RE}(\frac{1}{2}+\varepsilon), |E(R)| \leq \beta n \log n} \mathbf{P}\Big(G_p \text{ does not complement } R \,|\, R \subset G_p \Big) \cdot \mathbf{P}(R \subset G_p), \qquad (4.1)$$

where the union (and sum) is taken over all graphs R on n vertices which have property $\mathcal{RE}(\frac{1}{2}+\varepsilon)$ and at most $\beta n \log n$ edges.

Let us first examine the term $\mathbf{P}\Big(G_p \text{ does not complement } R \,|\, R \subset G_p \Big)$. Let R be a fixed graph with property $\mathcal{RE}(\frac{1}{2}+\varepsilon)$ and P be a fixed path on the same vertex set containing some fixed edge e. The number of such paths is at most $n^2 \cdot n!$, since there are at most n^2 choices for the length of path P and the fixed edge e, and there are at most $n(n-1)\cdots(n-i+1)$ paths of length i, $1 \leq i \leq n$. If in $R \cup P$ there is a path longer than P containing e, then the condition of Definition 3.2 is already satisfied. Therefore we can assume that there is no such path in $R \cup P$. Then, by the definition of property $\mathcal{RE}(\frac{1}{2}+\varepsilon)$, we can find a set S_P and for every $v \in S_P$ a corresponding set T_v, both of size $(\frac{1}{2}+\varepsilon)n$, such that for every $w \in T_v$, there exists a path containing e of the same length as P in $R \cup P$ which starts at v and ends at w. If there exist vertices $v \in S_P$ and $w \in T_v$ such that $\{v, w\}$ is an edge of R, then this edge is also in $R \cup G_p$ and again Definition 3.2 is satisfied. If there are no such edges of R, then conditioned on $R \subset G_p$, each such pair of vertices is an edge in G_p independently with probability p. By the minimum degree condition on G, we have

$$e_G(S_P, T_v) \geq \varepsilon n \cdot \left(\frac{1}{2} + \varepsilon \right) n \geq \frac{\varepsilon n^2}{2}.$$

Therefore there are at least $\frac{1}{2} e_G(S_P, T_v) \geq \frac{\varepsilon n^2}{4}$ edges of G that we would like to be present in the graph (the factor $\frac{1}{2}$ comes from the fact that a same edge can be counted twice). If G_p does not complement the graph R, then a.a.s. there exists some path P such that no such edge appears in G_p. For a fixed path P, the probability of this event is at most $(1-p)^{\varepsilon n^2/4}$. Consequently, by taking the union bound over all choice of paths P, we see that for large enough $C = C(\varepsilon)$ and $p \geq \frac{C \log n}{n}$

$$\mathbb{P}\Big(G_p \text{ does not complement } R \,|\, R \subset G_p \Big) \leq n^2 \cdot n! \cdot (1-p)^{-\varepsilon n^2/4} \leq e^{-\varepsilon n^2 p/8}.$$

Therefore in (4.1), the right hand side can be bounded by

$$\mathbf{P} \leq e^{-\varepsilon n^2 p/8} \cdot \sum_{R \in \mathcal{RE}(\frac{1}{2}+\varepsilon), |E(R)| \leq \beta n \log n} \mathbf{P}(R \subset G_p).$$

Also note that for a fixed graph R with t edges $\mathbf{P}(R \subset G_p) \leq \mathbf{P}(R \subset G(n,p)) = p^t$. Therefore, by taking the sum over all possible graphs R with at most $\beta n \log n$ edges, we can bound the probability that the assertion of the lemma fails by

$$\mathbf{P} \leq e^{-\varepsilon n^2 p/8} \sum_{t=1}^{\beta n \log n} \binom{\binom{n}{2}}{t} p^t \leq e^{-\varepsilon n^2 p/8} \sum_{t=1}^{\beta n \log n} \left(\frac{e n^2 p}{t} \right)^t.$$

Since $p \geq \frac{C \log n}{n}$, for $C \geq \beta$, the summands are monotone increasing in the range $1 \leq t \leq \beta n \log n$, and thus we can take the case $t = \beta n \log n$ for an upper bound on every term. This gives

$$\mathbf{P} \leq (1 + o(1)) \beta n \log n \cdot \left(e^{-\varepsilon np/(8\beta \log n)} \cdot \left(\frac{enp}{\beta \log n} \right) \right)^{\beta n \log n},$$

which is $o(1)$ for sufficiently large C depending on ε and β, since $p \geq \frac{C \log n}{n}$. This completes the proof. $\qquad \square$

Theorem 4.4. *There exists a constant C such that the following holds for every $p \geq \frac{C \log n}{n}$. If G is a Dirac graph satisfying (i) of Lemma 2.1, then G_p is a.a.s. Hamiltonian.*

Proof. Let $\varepsilon = \varepsilon_{4.2}$, $\beta = \beta_{4.2}$, and $C = \max\{\beta, C_{4.3}(\varepsilon, \beta)\}$. By Lemma 4.2, we know that G_p a.a.s. contains a subgraph that has property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ and at most $\beta n \log n$ edges. Then by Lemma 4.3, we know that G_p a.a.s. complements this subgraph. Therefore when both events hold, we see by Proposition 3.3 that G_p is a.a.s. Hamiltonian. $\qquad \square$

4.2 Second case

Let G be a Dirac graph satisfying the following as in (ii) of Lemma 2.1: there exists a set A of size $\frac{n}{2} \leq |A| \leq (\frac{1}{2} + 16\alpha)n$ such that $e(A, \overline{A}) \leq 6\alpha n^2$, and the induced subgraphs on both A and $\overline{A}$ have minimum degree at least $\frac{n}{5}$. Let $k = |A| - |\overline{A}|$ so that $|A| = \frac{n+k}{2}$, $|\overline{A}| = \frac{n-k}{2}$, and $k \leq 32\alpha n$.

Note that
$$|A|\frac{n}{2} \leq e(A, V(G)) = e(A, A) + e(A, \overline{A}) \leq 2e(A) + 6\alpha n^2.$$

Therefore,

$$
\begin{aligned}
e(A) \geq & \frac{1}{2} \left(\frac{|A|n}{2} - 6\alpha n^2 \right) = \frac{1}{2} \left(|A|^2 - \left(|A| - \frac{n}{2} \right) |A| - 6\alpha n^2 \right) \\
\geq & \binom{|A|}{2} - \frac{k|A|}{4} - 3\alpha n^2 \geq \binom{|A|}{2} - 11\alpha n^2.
\end{aligned}
\tag{4.2}
$$

Similarly, we can show that $e(\overline{A}) \geq \binom{|\overline{A}|}{2} - 6\alpha n^2$.

Lemma 4.5. *There exists a positive real β such that the following holds for every $p \geq \frac{\beta \log n}{n}$: each of the graphs $G[A]_p$ and $G[\overline{A}]_p$ a.a.s. contains a subgraph with property $\mathcal{RE}(\frac{1}{6})$ that has at most $\beta n \log n$ edges.*

Proof. We will only verify the property for $G[A]_p$ since the property for $G[\overline{A}]_p$ can be verified similarly. Let β be a large enough constant. Given $p \geq \frac{\beta \log n}{n}$, let $p' = \frac{\beta \log n}{n}$ and let H be a random subgraph of $G[A]_p$ obtained by taking every edge of $G[A]_p$ independently with probability $\frac{p'}{p}$ (which is less than 1). Then H has the same distribution as $G[A]_{p'}$. Since $G[A]_{p'}$ a.a.s. has at most $\beta n \log n$ edges, it suffices to show that H a.a.s. has property $\mathcal{RE}(\frac{1}{6})$. By Lemma 3.16, it suffices to verify that H is an expander with parameters $\frac{1}{4}$ and 2^{16}. Equivalently, we need to verify the following two properties:

(i) For every subset of vertices X of size $|X| \leq \frac{|A|}{2^{24}}$, we have $|N(X)| \geq 2^{16}|X|$, and

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(ii) for every subset of vertices X of size $|X| \geq \frac{|A|}{2^{12}}$, we have $|N(X)| \geq \frac{3}{4}|A|$.

We claim that H a.a.s. has the following properties: (a) minimum degree is at least $\frac{np'}{6}$, (b) for every pair of sets X and Y of sizes $|X| \leq \frac{|A|}{2^{24}}$ and $|Y| < 2^{16}|X|$, we have $e_H(X,Y) < \frac{|X|np'}{10}$, and (c) for every pair of sets X and Y of sizes $|X| = \frac{|A|}{2^{12}}$ and $|Y| = \frac{1}{4}|A|$, $e_H(X,Y) > 0$. Suppose that H indeed satisfies these properties. For every set X of size $|X| \leq \frac{|A|}{2^{24}}$, if $|N(X)| < 2^{16}|X|$, then by (a) and (b) we will have

$$\frac{|X|np'}{10} > e_H(X,N(X)) = e_H(X,V(H)) \geq |X|\frac{np'}{6},$$

which is a contradiction. Thus (i) holds. One can also easily see that (c) implies (ii).

We omit the proof of Property (a) (we need β to be large enough for (a)) and verify Property (b). Let t be a positive integer satisfying $t \leq \frac{|A|}{2^{24}}$. For a fixed pair of sets X and Y of sizes $|X| = t$ and $|Y| = 2^{16}t$, in order to have $e_H(X,Y) \geq \frac{|X|np'}{10}$, at least $\frac{|X|np'}{20}$ non-ordered pairs that have one endpoint in X and the other in Y must be present. Thus the probability of the event $e_H(X,Y) \geq \frac{|X|np'}{10}$ is at most

$$\binom{|X||Y|}{\frac{|X|np'}{20}} \cdot (p')^{\frac{|X|np'}{20}} \leq \left(\frac{20e|Y|}{n}\right)^{|X|np'/20}.$$

Therefore, by taking the union bound over all choices of t and sets X,Y, we see that the probability of (b) being false is

$$\sum_{t=1}^{|A|/2^{24}} \binom{n}{t}\binom{n}{2^{16}t}\left(\frac{20 \cdot 2^{16}et}{n}\right)^{tnp'/20}$$

$$\leq \sum_{t=1}^{|A|/2^{24}} \left(\frac{en}{2^{16}t}\right)^{2^{17}t}\left(\frac{20 \cdot 2^{16}et}{n}\right)^{tnp'/20} = \sum_{t=1}^{|A|/2^{24}} \left(\left(\frac{en}{2^{16}t}\right)^{2^{17}}\left(\frac{20 \cdot 2^{16}et}{n}\right)^{np'/20}\right)^{t}$$

$$= \sum_{t=1}^{|A|/2^{24}} \left(\left(\frac{e}{2^{16}}\right)^{2^{17}} \cdot (20 \cdot 2^{16})^{2^{17}}\left(\frac{20 \cdot 2^{16}et}{n}\right)^{(np'/20)-2^{17}}\right)^{t}.$$

Since $t \leq \frac{n}{2^{24}}$, the summand is maximized at $t = 1$, and thus by $p' = \frac{\beta \log n}{n}$, the probability above can be bounded by

$$n \cdot (1 + o(1))n^{-\Omega(\log n)} = o(1).$$

To verify Property (c), let X and Y be fixed sets of size $|X| = \frac{|A|}{2^{12}}$ and $|Y| = \frac{1}{4}|A|$. Then since $e_G(A) \geq \binom{|A|}{2} - 11\alpha n^2$ (see (4.2)), we have $e_G(X,Y) \geq |X||Y| - 22\alpha n^2 \geq 2^{-15}|A|^2 \geq 2^{-17}n^2$. Thus by Chernoff's inequality, the probability that $e_H(X,Y) = 0$ is at most $e^{-\Omega(n^2p')} = e^{-\Omega(\beta n \log n)}$. By taking the union bound over all pairs of sets X and Y, we obtain (c). □

Lemma 4.6. *For every fixed positive real β, there exists a constant $C = C(\beta)$ such that the following holds for every $p \geq \frac{C \log n}{n}$: each of the graphs $G[A]_p$ and $G[\overline{A}]_p$ a.a.s. complements its every subgraph with property $\mathcal{RE}(\frac{1}{6})$ that has at most $\beta n \log n$ edges.*

Proof. It suffices to make a slight modification to the proof of Lemma 4.3 to prove this lemma. As in Lemma 4.5, we only consider the graph $G[A]$. Fix a graph R with property $\mathcal{RE}(\frac{1}{6})$ that has at most $\beta n \log n$ edges, and let P be a maximum path in $R \cup P$. Let S_P and T_v be given as in Definition 3.1. Then since $e_G(A) \geq \binom{|A|}{2} - 11\alpha n^2$ (see (4.2)), we have

$$e_G(S_P, T_v) \geq |S_P| \cdot |T_v| - 22\alpha n^2 \geq \frac{|A|^2}{36} - 22\alpha n^2 \geq \frac{n^2}{200}.$$

We can proceed exactly as in Lemma 4.3 to conclude our lemma. $\qquad\square$

Theorem 4.7. *There exists a constant C such that the following holds for every $p \geq \frac{C \log n}{n}$. If G is a Dirac graph satisfying (ii) of Lemma 2.1, then G_p is a.a.s. Hamiltonian.*

Proof. Let $\beta = \beta_{4.5}$ and $C = \max\{\beta, C_{4.6}(\beta), 4\}$. By Lemma 4.5, we know that a.a.s. each $G_p[A]$ and $G_p[\overline{A}]$ contains a subgraph that has property $\mathcal{RE}(\frac{1}{6})$ and at most $\beta n \log n$ edges. Then by Lemma 4.6, we know that $G_p[A]$ and $G_p[\overline{A}]$ complement each of their subgraphs with the above property. Therefore when both events hold, we see by Proposition 3.3 that $G_p[A]$ and $G_p[\overline{A}]$ are Hamiltonian connected. It then suffices to show that in G_p a.a.s. there exist two vertex disjoint edges that connect A and $\overline{A}$, since together with the Hamilton connectivity of $G_p[A]$ and $G_p[\overline{A}]$ this will imply Hamiltonicity of G_p. In order to prove this, consider the bipartite graph $\mathcal{B}$ induced by the edges of G between A and $\overline{A}$, and let $\mathcal{B}_p = \mathcal{B} \cap G_p$. By Hall's theorem, it suffices to show that $\mathcal{B}_p$ a.a.s. does not have a vertex that dominates all the edges of $\mathcal{B}_p$.

Let v be a fixed vertex and first assume that $|A| = |\overline{A}| = \frac{n}{2}$. Since the minimum degree of G is at least $\frac{n}{2}$, the graph $\mathcal{B}$ has minimum degree at least 1. Thus there exist at least $\frac{n}{2} - 1$ edges which are not incident to v, and the probability that all the edges of $\mathcal{B}_p$ are incident to v is at most $(1-p)^{n/2-1} \leq e^{-C \log n/3}$. If $|A| = \frac{n}{2} + t$ for some $t > 0$, then all the vertices of $\overline{A}$ have degree at least $\lceil t + 1 \rceil \geq 2$ in $\mathcal{B}$. Since $t \leq 16\alpha n$, the total number of edges in $\mathcal{B}$ is at least $2 \cdot (\frac{n}{2} - t) \geq (1 - 32\alpha)n$, and since the maximum degree of $\mathcal{B}$ is at most $|A| \leq (\frac{1}{2} + 16\alpha)n$, the number of edges not incident to v is at least $(1 - 32\alpha)n - (\frac{1}{2} + 16\alpha)n \geq \frac{n}{3}$. Therefore in this case, the probability that all the edges of $\mathcal{B}_p$ are incident to v is at most $(1-p)^{n/3} \leq e^{-C \log n/3}$.

Thus in either of the cases, for a fixed vertex v, the probability that v dominates all the edges of $\mathcal{B}_p$ is at most $e^{-C \log n/3}$. Since $C \geq 4$, this probability is $o(n^{-1})$, and by taking the union bound over all the vertices, we can conclude that a.a.s. there is no vertex which is incident to all the edges of $\mathcal{B}_p$. This concludes the proof. $\qquad\square$

4.3 Third case

Let G be a Dirac graph satisfying the following as in (iii) of Lemma 2.1. There exists a set A of size $\frac{n}{2} \leq |A| \leq (\frac{1}{2} + 16\alpha)n$ such that the bipartite graph induced by the edges between A and $\overline{A}$ has at least $(\frac{1}{4} - 14\alpha)n^2$ edges and minimum degree at least $\frac{n}{64}$. Moreover, either $|A| = \lceil \frac{n}{2} \rceil$, or the induced subgraph $G[A]$ has maximum degree at most $\frac{n}{32}$. Let $V_1 = A$ and $V_2 = \overline{A}$. Let $k = |V_1| - |V_2|$ so that $|V_1| = \frac{n+k}{2}$, $|V_2| = \frac{n-k}{2}$, and $k \leq 32\alpha n$.

Lemma 4.8. *There exists a constant β such that the following holds for every $p \geq \frac{\beta \log n}{n}$: G_p a.a.s. contains a subgraph with property $\mathcal{RE}_b(\frac{1}{6})$ that has at most $\beta n \log n$ edges.*

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Proof. Let $\beta \geq 128$ be a large enough constant. Let $p' = \frac{\beta \log n}{n}$ and note that $p \geq p'$. Let H be a subgraph of G_p obtained by taking each edge independently with probability $\frac{p'}{p}$, and note that the distribution of H is identical to that of $G_{p'}$. Since $G_{p'}$ a.a.s. has at most $(1 + o(1))p' \cdot e(G) \leq \beta n \log n$ edges, it suffices to show that $G_{p'}$ a.a.s. has property $\mathcal{RE}_b(\frac{1}{6})$.

We first show that $G_{p'}$ contains at least k vertex disjoint edges in V_1. To show this, it suffices to show that $G_{p'}[V_1]$ a.a.s. has covering number at least $2k - 1$. Since this is trivial for $k = 0$, we may assume that $k \geq 1$. By the union bound, we can bound the probability that the covering number is at most $2k - 2$ as follows:

$$\sum_{X \subset V_1, |X| = 2k-2} \mathbf{P}(V_1 \setminus X \text{ is an independent set in } G_{p'}). \tag{4.3}$$

Note that since $|V_2| = \frac{n-k}{2}$, the induced subgraph $G[V_1]$ must have minimum degree at least $\frac{k}{2}$. Moreover, since the graph $G[V_1]$ has maximum degree at most $\frac{n}{32}$, we can see that for a set X of size $|X| = 2k - 2$, the number of edges of G in $V_1 \setminus X$ is at least

$$\frac{1}{2} \cdot \frac{k}{2} \cdot |V_1| - (2k-2)\frac{n}{32} \geq \frac{kn}{8} - \frac{kn}{16} = \frac{kn}{16},$$

and therefore

$$\mathbf{P}(V_1 \setminus X \text{ is an independent set}) \leq (1-p)^{kn/16} \leq e^{-knp/16} \leq n^{-\beta k/16}.$$

By using this inequality (note that $\beta \geq 128$), we can bound (4.3) from above by

$$\binom{n}{2k} \cdot n^{-\beta k/16} = o(1).$$

Consequently we a.a.s. have k vertex disjoint edges in $G_{p'}[V_1]$. Condition on this event being true. Arbitrarily pick k vertex disjoint edges in V_1 as our special edges S_E, and for each such edge, declare one of its vertices as a special vertex (let S_V be the set of special vertices). Note that $G_{p'}$ contains a special frame (V_1, V_2, S_V, S_E). Let V_1', V_1'' be as in the Definition 3.8. We will prove that for large enough β, conditioned on the special frame, $G_{p'}$ a.a.s. has the following two properties:

1. For every $X \subset V_1$, if $|X| \leq \frac{n}{2^{30}}$, then $|N(X) \cap V_2| \geq 2^{20}|X|$, and if $|X| \geq \frac{n}{2^{15}}$, then $|N(X) \cap V_2| \geq \frac{3}{4}|V_2|$, and

2. for every $Y \subset V_2$, if $|Y| \leq \frac{n}{2^{30}}$, then $|N(Y) \cap V_1''| \geq 2^{20}|Y|$, and if $|Y| \geq \frac{n}{2^{15}}$, then $|N(Y) \cap V_1''| \geq \frac{3}{4}|V_1''|$.

This will be done by establishing the following properties that the bipartite subgraph H of $G_{p'}$ containing all the edges of $G_{p'}$ between V_1, V_2 a.a.s. has: (a) minimum degree is at least $\frac{np'}{70}$, (b) for every pair of sets X and Y of sizes $|X| \leq \frac{n}{2^{30}}$ and $|Y| < 2^{20}|X|$, we have $e_H(X, Y) < \frac{|X|np'}{80}$, and (c) for every pair of sets X and Y of sizes $|X| = \frac{n}{2^{15}}$ and $|Y| = \frac{n}{10}$, $e_H(X, Y) > 0$. To verify (a), we can use the fact that the bipartite graph induced by the edges between A and $\overline{A}$ has minimum degree at least $\frac{n}{64}$. The proof of (b) follows from direct application of Chernoff's inequality and the union bound. To verify (c), we can use that $e(A, \overline{A}) \geq (\frac{1}{4} - 14\alpha)n^2 \geq |A| \cdot |\overline{A}| - 14\alpha n^2$ and

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therefore $e(X,Y) \geq |X||Y| - 14\alpha n^2 \geq 2\alpha n^2$. Detailed computation and the derivation of (i), (ii) from (a), (b), and (c) are similar to that of Lemma 4.2. Once we verify these properties we have a bipartite-expander with parameters $\frac{1}{4}$ and 2^{30} (see Definition 3.13), and by Lemma 3.14, we can derive that our graph has property $\mathcal{RE}_b(\frac{1}{6})$. $\qquad\square$

Lemma 4.9. *For every fixed positive real β, there exists a constant $C = C(\beta)$ such that the following holds for every $p \geq \frac{C \log n}{n}$: G_p a.a.s. complements every its subgraph with property $\mathcal{RE}_b(\frac{1}{6})$ that has at most $\beta n \log n$ edges.*

Proof. As in the proof of Lemma 4.6, it suffices to make a slight modification to the proof of Lemma 4.3 to prove this lemma. Fix a graph R with property $\mathcal{RE}(\frac{1}{6})$ that has at most $\beta n \log n$ edges, and let P be a maximum path in $R \cup P$. Let S_P and T_v be given as in Definition 3.1. Then since $e_G(A, \overline{A}) \geq \frac{n^2}{4} - 14\alpha n^2 \geq |A| \cdot |\overline{A}| - 14\alpha n^2$, we have

$$e_G(S_P, T_v) \geq |S_P| \cdot |T_v| - 14\alpha n^2 \geq \frac{n^2}{36} - 14\alpha n^2 \geq \frac{n^2}{40}.$$

We can proceed exactly as in Lemma 4.3 to conclude our lemma. $\qquad\square$

Theorem 4.10. *There exists a constant C such that the following holds for every $p \geq \frac{C \log n}{n}$. If G is a Dirac graph satisfying (iii) of Lemma 2.1, then G_p is a.a.s. Hamiltonian.*

Proof. Let $\beta = \beta_{4.8}$, and let $C = \max\{\beta, C_{4.9}(\beta)\}$. By Lemma 4.8, we know that G_p a.a.s. contains a subgraph that has property $\mathcal{RE}_b(\frac{1}{6})$ and at most $\beta n \log n$ edges. Then by Lemma 4.9, we know that G_p a.a.s. complements this subgraph. Therefore when both events hold, we see by Proposition 3.12 that G_p is Hamiltonian. $\qquad\square$

5 Hamiltonicity game on Dirac graphs

In this section we prove Theorem 1.2. We begin by presenting some standard results and techniques in positional game theory which we will need later. In 1973, Erdős and Selfridge [18] gave a sufficient condition for Breaker's win in a $(1 : 1)$ Maker-Breaker game. Later, Beck [5] generalized this result and proved the following theorem.

Theorem 5.1. *Consider a $(p : q)$ Maker-Breaker game played over some board where $\mathcal{F}$ is the family of winning sets. If*

$$\sum_{B \in \mathcal{F}} (1+q)^{-|B|/p} < \frac{1}{1+q},$$

then Breaker has a winning strategy.

Suppose that we are playing a $(1 : b)$ Maker-Breaker game over a board V with the family $\mathcal{F}$ of winning sets, and let a be some fixed integer. It is sometimes convenient to partition the board into a boards $V_1 \cup \cdots \cup V_a$ and to play a $(1 : ab)$ game on each board V_i separately. That is, Maker will start by playing the board V_1, and after playing board V_i, in the next round will play board V_{i+1} (index addition is modulo a). Note that after Maker plays the board V_1 for the first time, and until playing

it for the second time, Breaker can claim at most ab elements of V_1. Therefore Maker may assume to his/her disadvantage that he/she is playing a $(1 : ab)$ game on each board as the second player. If one shows that Maker can claim elements so that certain properties are satisfied for each board, then by combining these properties, we may show in the end that the Maker's elements altogether contain some winning set. When we say that we *split the board*, we suppose that we partitioned the board into some fixed number of boards as above.

We will use later the following concentration result (see, e.g., [19, Theorem 2.10]). Let A and A' be sets such that $A' \subset A$, and $|A| = N$, $|A'| = m$. Let B be a subset of A of size n chosen uniformly at random. Then the distribution of the random variable $|B \cap A'|$ is called the *hypergeometric distribution with parameters N, n, and m*.

Theorem 5.2. *Let ε be a fixed positive constant and let X be a random variable with hypergeometric distribution with parameters N, n, and m. Then for all $t \geq 0$,*

$$P\big(|X - \mathbb{E}[X]| \geq t\big) \leq 2e^{-2t^2/n}.$$

Let G be a Dirac graph, and as in the previous section, we begin by applying Lemma 2.1 with $\alpha = \frac{1}{2^{40}}$ and $\gamma = \frac{1}{96}$ to classify Dirac graphs into three types. We will show case by case that Maker can win a $(1 : b)$ Hamiltonicity Maker-Breaker game played on G if $b \leq \frac{cn}{\log n}$ for some small positive constant c.

5.1 First case

We first assume that $e(A, B) \geq \alpha n^2$ for all half-sets A and B as in (i) of Lemma 2.1.

Lemma 5.3. *There exist positive reals ε, c, and β the following holds for every $b \leq \frac{cn}{\log n}$. In a $(1 : b)$ Maker-Breaker game played on G, Maker can construct a graph M_1 with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ in the first $\beta n \log n$ rounds.*

Proof. Let $\varepsilon = \min\{\varepsilon_{3.5}, \left(\frac{\alpha}{8}\right)^5\}$ and $C = 16\varepsilon^{-3}$. Let $c = \min\{\frac{1}{42C}, \frac{\alpha}{30}\}$, $\beta = \frac{1}{2c}$, and $b_0 = \frac{cn}{\log n}$. We will show that in a $(1 : b_0)$ Maker-Breaker game played on G, Maker can construct a graph with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$. If this indeed is true, then since at most $\beta n \log n$ rounds can be played, we can see that for all $b \leq b_0$, in a $(1 : b)$ Maker-Breaker game played on G, Maker can construct a graph M_1 with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ in the first $\beta n \log n$ rounds. This is because Maker can always trick himself/herself that he/she is playing a $(1 : b_0)$ game, by arbitrarily adding $b_0 - b$ 'fake' Breaker edges. If Breaker later happens to claim some fake Breaker edge, then Maker will replace that fake edge with another fake edge which has not yet been claimed. In this way, Maker is essentially playing a $(1 : b_0)$ game, and thus can construct a graph M_1 with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ in the first $\binom{n}{2}/b_0 \leq \beta n \log n$ rounds.

Let $r = C \log n$ and note that by Lemma 3.5, it suffices to show that Maker can construct a half-expander with parameters ε and r. A naive approach directly using Beck's criteria (Theorem 5.1) fails for our range of bias, and thus we use the strategy of Krivelevich and Szabó [26]. In this strategy, we construct an auxiliary hypergraph whose vertex set is the edge set of G, and play a Maker-Breaker game on this hypergraph. The board is the vertex set of the hypergraph, and the winning sets are the edges of the hypergraph (to avoid confusion, we name the players of this game

as NewMaker and NewBreaker). Maker (resp. Breaker) of the original game will play NewBreaker (resp. NewMaker) in the auxiliary game. By doing so, we wish to establish the fact that Maker can strategically claim edges so that Maker's graph satisfies the conditions of a half-expander.

For each $\ell = 1, \cdots, n$, let $V_{\ell,1}, \cdots, V_{\ell,\ell}$ be fixed disjoint vertex subsets of $V(G)$ of size $\lfloor \frac{n}{\ell} \rfloor$, and for each index subset $J \subset [\ell]$, let $V_{\ell,J} = \cup_{j \in J} V_{\ell,j}$. Let H_1, H_2 and H_3 be hypergraphs that have the edge set of G as their vertex set. The edge set of H_1 is constructed as follows: for each set of vertices $X \subset V(G)$ of size $|X| = i \leq \frac{\varepsilon n}{r}$ and each index set $J \subset [3ri]$ of size $|J| = 2ri$, place a hyperedge consisting of the edges of G that have one endpoint in X, and the other endpoint in $V_{3ri,J} \setminus X$. Since G has minimum degree at least $\frac{n}{2}$, the size of a hyperedge constructed this way is at least

$$|X| \cdot \left(|J| \cdot |V_{3ri,1}| - \frac{n}{2} - |X| \right) \geq i \cdot \left(2ri \left(\frac{n}{3ri} - 1 \right) - \frac{n}{2} - i \right) \geq \left(\frac{1}{6} - o(1) \right) ni \geq \frac{ni}{7},$$

and the number of hyperedges of H_1 constructed from subsets of vertices of size i is $\binom{n}{i}\binom{3ri}{2ri}$. Assume that Maker claims at least one edge (vertex of the hypergraph) in each of the hyperedges as above. Then it follows that for each set of vertices X of size at most $\frac{\varepsilon n}{r}$, Maker's graph has $|N(X)| > ri = r|X|$, as otherwise, we can find an index set $J \subset [3ri]$ of size $|J| = 2ri$ such that there are no edges which have one endpoint in X and the other endpoint in $V_{3ri,J}$. Thus in such a situation, Maker's graph will satisfy Condition (i) of Definition 3.4.

The edge set of H_2 is constructed as follows: for each pair of sets of vertices $X, Y \subset V(G)$ of sizes $|X| = \frac{n}{\varepsilon r}$ and $|Y| = (\frac{1}{2} + \varepsilon)n$, place a hyperedge consisting of the edges of G that have one endpoint in X, and the other endpoint in $Y \setminus X$. Since G has minimum degree at least $\frac{n}{2}$, the size of a hyperedge constructed this way is at least

$$|X| \cdot \left(|Y| - \frac{n}{2} - |X| \right) \geq \frac{n}{\varepsilon r} \cdot \left(\varepsilon n - \frac{n}{\varepsilon r} \right) \geq \frac{n^2}{2r},$$

and the number of such hyperedges is at most 2^{2n}. Moreover, if Maker can claim at least one edge in each of the hyperedges of H_2, then Maker's graph will satisfy Condition (ii) of Definition 3.4.

For every pair of disjoint sets of vertices X and Y such that $|X|, |Y| \geq (\frac{1}{2} - \varepsilon^{1/5})n$, let $E_{X,Y}$ be the set of edges that have one endpoint in X and the other endpoint in Y (edges within $X \cap Y$ are counted once). The edge set of H_3 is as follows: for every X and Y as above, place a hyperedge over every subset of $E_{X,Y}$ of size exactly $|E_{X,Y}| - 2n$. Since $\frac{3}{4}\alpha n^2 \leq \alpha n^2 - 2\varepsilon^{1/5}n^2 \leq e_G(X,Y) \leq n^2$, each such hyperedge has size at least $|E_{X,Y}| \geq \frac{1}{2}e_G(X,Y) - 2n \geq \frac{\alpha n^2}{3}$, and the total number of hyperedges of H_3 is at most

$$2^{2n} \cdot \binom{n^2}{2n} \leq 2^{2n} \cdot n^{4n} \leq e^{5n \log n}.$$

Moreover, if Maker can claim at least one edge in each of the hyperedges of H_3, then Maker's graph will satisfy Condition (iii) of Definition 3.4.

Let $H = H_1 \cup H_2 \cup H_3$ and consider a $(b_0 : 1)$ Maker-Breaker game played on the hypergraph H (where we name the players as NewMaker and NewBreaker in order to distinguish the players from our game). By the arguments above, it suffices to show that the auxiliary game is NewBreaker's win in order to establish our lemma. This will be done by using Theorem 5.1 with $p = b_0$ and $q = 1$. We

thus would like to show that

$$\sum_{i=1}^{3} \sum_{e \in E(H_i)} 2^{-|e|/b_0} = \sum_{i=1}^{3} \sum_{e \in E(H_i)} 2^{-|e| \log n/(cn)} \tag{5.1}$$

is at most $\frac{1}{2}$. For the hypergraph H_1, using the fact $c \leq \frac{1}{42C}$, we have

$$\sum_{e \in E(H_1)} 2^{-|e|/b_0} \leq \sum_{i=1}^{\varepsilon n/r} \binom{n}{i} \binom{3ri}{2ri} 2^{-ni \log n/(7cn)} \leq \sum_{i=1}^{\varepsilon n/r} n^i \cdot 2^{3ri} \cdot 2^{-i \log n/(7c)}$$

$$\leq \sum_{i=1}^{\varepsilon n/r} e^{(3C+1-\frac{1}{7c})i \log n} \leq \frac{1}{4}.$$

For hypergraphs H_2 and H_3, using the fact $c \leq \min\{\frac{1}{42C}, \frac{\alpha}{30}\}$, we have

$$\sum_{e \in E(H_2)} 2^{-|e|/b_0} + \sum_{e \in E(H_3)} 2^{-|e|/b_0} \leq 2^{2n} \cdot 2^{-n^2 \log n/(2rcn)} + e^{5n \log n} \cdot 2^{-\alpha n^2 \log n/(3cn)}$$

$$\leq 2^{2n} \cdot 2^{-n/(2cC)} + e^{5n \log n} \cdot 2^{-\alpha n \log n/(3c)}$$

$$\leq \frac{1}{4}.$$

Therefore, by combining the two inequalities, we get $(5.1) \leq \frac{1}{2}$. $\qquad\square$

Lemma 5.4. *For every fixed positive reals ε and β, there exists a positive constant $c = c(\varepsilon, \beta)$ satisfying the following for every $b \leq \frac{cn}{\log n}$. In a $(1 : b)$ Maker-Breaker game played on G, if Maker constructs a graph M_1 with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ in the first $\beta n \log n$ rounds, then he in the remaining rounds can construct a graph M_2 which complements M_1.*

Proof. Let $c \leq \min\{c_{5.3}(\varepsilon), \frac{\varepsilon^2}{2\beta}, \frac{\varepsilon}{8}\}$. Note that since we played $\beta n \log n$ rounds to construct M_1, the number of edges claimed so far is at most $c\beta n^2 + \beta n \log n \leq 2c\beta n^2 \leq \varepsilon^2 n^2$. Let P be a path with a fixed edge e such that there is no path longer than P containing e in the graph $P \cup M_1$. Then there exists a set $S_P \subset V(P)$ of size $|S_P| \geq (\frac{1}{2} + \varepsilon)n$ such that for every $v \in S_P$ there exists a set T_v of size $|T_v| \geq (\frac{1}{2} + \varepsilon)n$ such that for all $w \in T_v$, there exists a path of the same length as P containing e, starting at v and ending at w.

Since G has minimum degree at least $\frac{n}{2}$, we know that for each vertex $v \in S_P$, at least εn vertices in T_v form an edge with v in the graph G. Since at most $\varepsilon^2 n^2$ edges have been claimed so far, in total we have at least

$$\frac{1}{2} \cdot \left(\left(\frac{1}{2} + \varepsilon \right) n \cdot \varepsilon n - \varepsilon^2 n^2 \right) \geq \frac{\varepsilon}{4} n^2$$

edges such that if Maker can claim at least one of these edges, then he can extend P. Consequently, if Maker can do this for all paths P, then we prove our lemma (the factor $\frac{1}{2}$ comes from the fact that the same pair (v, w) can be counted at most twice, once as $v \in S_P, w \in T_w$ and once as $w \in S_P, v \in T_w$).

There are at most $n^2 \cdot n!$ paths that we need to consider, and for each path we have $\frac{\varepsilon}{4} n^2$ edges where Maker has to claim at least one of these edges. Consider the following Maker-Breaker game

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(where we name the players as NewMaker and NewBreaker in order to distinguish the players from our game). The board is defined as the edges which have not been claimed in the first $\beta n \log n$ rounds. The winning sets are defined as sets of at least $\frac{\varepsilon}{4} n^2$ edges for each non-extendable path P with a fixed edge which we described above. Note that there are at most $n^2 \cdot n!$ winning sets. It suffices to show that NewBreaker wins this new game, since our Maker will play NewBreaker's role here (thus he/she wants to claim at least one edge from each of the winning sets). We can use Beck's criterion, Theorem 5.1, with $p = \frac{cn}{\log n}$ and $q = 1$ to see that the newly defined game is indeed NewBreaker's win since $c \leq \frac{\varepsilon}{8}$:

$$\sum_{B \in \mathcal{F}} 2^{-|B|/p} \leq n^2 \cdot n! \cdot 2^{-(\varepsilon/4)n^2/(cn/\log n)} < e^{n \log n} 2^{-(\varepsilon/(4c))n \log n} < \frac{1}{2}.$$

$\square$

Theorem 5.5. *There exists a constant c such that the following holds for every $b \leq \frac{cn}{\log n}$. If G is a Dirac graph satisfying (i) of Lemma 2.1, then Maker has a winning strategy for the $(1 : b)$-Maker-Breaker Hamiltonicity game.*

Proof. Let $\varepsilon = \varepsilon_{5.3}$, $\beta = \beta_{5.3}$ and let $c = \min\{c_{5.3}, c_{5.4}(\varepsilon, \beta)\}$. By Lemma 5.3, Maker can construct a graph M_1 with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ in the first $\beta n \log n$ rounds. Then by Lemma 5.4, Maker can construct a graph M_2 which complements G_1 in the remaining rounds. Therefore by Proposition 3.3, Maker can construct a Hamilton cycle and win the game. $\square$

5.2   Second case

We assume that there exists a set A of size $\frac{n}{2} \leq |A| \leq (\frac{1}{2} + 16\alpha)n$ such that $e(A, \overline{A}) \leq 6\alpha n^2$, and the induced subgraphs on both A and $\overline{A}$ have minimum degree at least $\frac{n}{5}$, as in (ii) of Lemma 2.1. The same computation as in (4.2) shows that

$$e(A) \geq \binom{|A|}{2} - 11\alpha n^2 \quad \text{and} \quad e(\overline{A}) \geq \binom{|\overline{A}|}{2} - 6\alpha n^2. \tag{5.2}$$

Lemma 5.6. *There exist positive reals c and β satisfying the following for every positive $b \leq \frac{cn}{\log n}$. In a $(1 : b)$ Maker-Breaker game played on the board $G[A]$, Maker can construct a graph with property $\mathcal{RE}(\frac{1}{6})$ in the first $\beta n \log n$ rounds (similar for $G[\overline{A}]$).*

Proof. Let c be a small enough constant depending on α. Let $\beta = \frac{1}{2c}$ and $b_0 = \frac{cn}{\log n}$. We will show that in a $(1 : b_0)$ Maker-Breaker game played on $G[A]$, Maker can construct a graph with property $\mathcal{RE}(\frac{1}{2} + \varepsilon)$ (similar proof can be used to establish the statement for $G[\overline{A}]$). As in Lemma 5.3, this will imply that for all $b \leq b_0$, in a $(1 : b)$ Maker-Breaker game played on G, Maker can construct a graph M_1 with property $\mathcal{RE}(\frac{1}{6})$ in the first $\beta n \log n$ rounds. Note that by Lemma 3.16, it suffices to show that Maker can construct a half-expander with parameters $\frac{1}{4}$ and $r = 2^{20}$. We will construct an auxiliary hypergraph whose vertex set is the edge set of G, and play a Maker-Breaker game on this hypergraph (we name the players of this game as NewMaker and NewBreaker). Maker (resp. Breaker) of the original game will play NewBreaker (resp. NewMaker) in the auxiliary game.

For each $\ell = 1, \cdots, n$, let $V_{\ell,1}, \cdots, V_{\ell,\ell}$ be fixed disjoint vertex subsets of $V(G)$ of size $\lfloor \frac{|A|}{\ell} \rfloor$, and for each index subset $J \subset [\ell]$, let $V_{\ell,J} = \cup_{j \in J} V_{\ell,j}$. Let H_1 and H_2 be hypergraphs that have the edge set of G as their vertex set. The edge set of H_1 is constructed as follows: for each set of vertices $X \subset V(G)$ of size $|X| = i \leq \frac{n}{r^{3/2}}$ and each index set $J \subset [10ri]$ of size $|J| = 9ri$, place a hyperedge consisting of the edges of G that have one endpoint in X, and the other endpoint in $V_{10ri,J} \setminus X$. Since G has minimum degree at least $\frac{n}{5}$, the size of a hyperedge constructed this way is at least

$$|X| \cdot \left(|J| \cdot |V_{10ri,1}| - (|A| - \frac{n}{5}) - |X| \right) \geq i \cdot \left(9ri \left(\frac{|A|}{10ri} - 1 \right) - \frac{4|A|}{5} - i \right) \geq \frac{|A|i}{11},$$

and the number of hyperedges of H_1 constructed from subsets of vertices of size i is $\binom{|A|}{i}\binom{10ri}{9ri}$. Assume that Maker claims at least one edge (vertex of the hypergraph) in each of the hyperedges as above. Then it follows that for each set of vertices X of size at most $\frac{|A|}{r^{3/2}}$, Maker's graph has $|N(X)| > ri = r|X|$, as otherwise, we can find an index set $J \subset [10ri]$ of size $|J| = 9ri$ such that there are no edges which have one endpoint in X and the other endpoint in $V_{10ri,J}$. Thus in such a situation, Maker's graph will satisfy Condition (i) of Definition 3.15.

The edge set of H_2 is constructed as follows: for each pair of sets of vertices $X, Y \subset V(G)$ of sizes $|X| = \frac{|A|}{r^{3/4}}$ and $|Y| = \frac{|A|}{4}$, place a hyperedge consisting of the edges of G that have one endpoint in X, and the other endpoint in $Y \setminus X$. By (5.2), the size of a hyperedge constructed this way is at least

$$|X| \cdot (|Y| - |X|) - 11\alpha n^2 \geq \frac{|A|}{2^{15}} \cdot \left(\frac{|A|}{4} - \frac{|A|}{2^{15}} \right) - 11\alpha n^2 \geq \frac{|A|^2}{2^{20}},$$

and the number of such hyperedges is at most 2^{2n}. Moreover, if Maker can claim at least one edge in each of the hyperedges of H_2, then Maker's graph will satisfy Condition (ii) of Definition 3.15.

Let $H = H_1 \cup H_2$ and consider a $(b_0 : 1)$ Maker-Breaker game played on the hypergraph H (where we name the players as NewMaker and NewBreaker in order to distinguish the players from our game). By the observations above, it suffices to show that the auxiliary game is NewBreaker's win in order to establish our lemma. This can be done by using Theorem 5.1 with $p = b_0$ and $q = 1$, given that c is small enough. We omit the detailed computation. $\square$

Lemma 5.7. *For every fixed positive real β, there exists a constant $c = c(\beta)$ such that the following holds for every positive $b \leq \frac{cn}{\log n}$. In a $(1 : b)$ Maker-Breaker game played on the board $G[A]$, suppose that Maker constructed a graph with property $\mathcal{RE}(\frac{1}{6})$ in the first $\beta n \log n$ rounds. Then he can construct a graph that complements it in the remaining rounds (similar for $G[\overline{A}]$).*

Proof. We will only prove the statement for $G[A]$, since the statement for $G[\overline{A}]$ can be proved similarly. Let $c \leq \alpha\beta^{-1}$ be a small enough constant. Let M_1 be the Maker's graph constructed in the first $\beta n \log n$ rounds. Note that the number of edges that have been claimed in the first $\beta n \log n$ rounds is at most $c\beta n^2 + \beta n \log n \leq 2c\beta n^2$. Let G' be the graph of the edges that have not been claimed by Breaker so far. Let P be a path over a subset of vertices of A, with a fixed edge e such that there is no path longer than P containing e in the graph $P \cup M_1$. Then there exists a set $S_P \subset P$ of size $|S_P| \geq \frac{|A|}{6}$ such that for every $v \in S_P$ there exists a set T_v of size $|T_v| \geq \frac{|A|}{6}$ such that for all $w \in T_v$, there exists a path of the same length as P containing e, starting at v and ending at w. By

(5.2), we have

$$e_{G'}(S_P, T_v) \geq |S_P| \cdot |T_v| - 11\alpha n^2 - 2c\beta n^2 \geq \frac{|A|^2}{36} - 13\alpha n^2 \geq \frac{n^2}{200}.$$

Using this estimate, we can proceed as in Lemma 5.4 to finish the proof. We omit the details. $\square$

By using the two lemmas above, we can show that Maker can win the Hamiltonicity game in this case as well.

Theorem 5.8. *There exists a constant c such that the following holds for every positive $b \leq \frac{cn}{\log n}$. If G is a Dirac graph satisfying (ii) of Lemma 2.1, then Maker has a winning strategy for the $(1 : b)$-Maker-Breaker Hamiltonicity game.*

Proof. Let $\beta = \beta_{5.6}$, and $c = \frac{1}{3}\min\{c_{5.6}, c_{5.7}(\beta)\}$. We split the board into three boards, $G[A], G[\overline{A}]$, and the subgraph induced by the edges between A and $\overline{A}$ (call the last one the bipartite board $\mathcal{B}$). To Maker's disadvantage, we will separately play a $(1 : 3b)$ game on each board, and show that Maker can play the three games so that in the end, the union of Maker's graphs in the three boards is Hamiltonian. By Lemma 5.6, we know that Maker can construct subgraphs of $G[A]$ and $G[\overline{A}]$ which have property $\mathcal{RE}(\frac{1}{6})$ in the first $\beta n \log n$ rounds of each board. Then by Lemma 5.7, Maker can construct graphs that complement these subgraphs in the remaining rounds of each board. Therefore, by Proposition 3.3, Maker can construct subgraphs of $G[A]$ and $G[\overline{A}]$ which are Hamiltonian connected. Thus it suffices to show that Maker can claim two vertex disjoint edges in the bipartite board, since together with the Hamilton connectivity of Maker's graph in $G[A]$ and $G[\overline{A}]$, this will imply Hamiltonicity of Maker's graph.

Consider the bipartite board $\mathcal{B}$. We will consider two cases depending on the sizes of A and $\overline{A}$. First assume that $|A| = |\overline{A}| = \frac{n}{2}$. Then since the minimum degree of G is at least $\frac{n}{2}$, the graph $\mathcal{B}$ has minimum degree at least 1. If there is no vertex in A of degree at least $\frac{n}{3}$ in $\mathcal{B}$, then Maker starts by claiming an arbitrary edge $\{v, w\}$ of $\mathcal{B}$ such that $v \in A, w \in \overline{A}$. Breaker can then claim at most $\frac{2cn}{\log n}$ other edges before Maker's next move since Breaker might have been the first player. Afterwards, since $\mathcal{B}$ has minimum degree at least 1, we can see that there exists at least $\frac{n}{2} - 1 - \frac{2cn}{\log n}$ vertices other than w in $\overline{A}$ which have at least 1 non-claimed edge incident to it. Among them, at most $\frac{n}{3}$ can be incident to v. Therefore Maker can claim an edge $\{v', w'\}$ such that $v \neq v'$ and $w \neq w'$. Similarly, we can take care of the case when $\overline{A}$ has no vertex of degree at least $\frac{n}{3}$. Thus we may assume that there exist vertices $v_0 \in A$ and $w_0 \in \overline{A}$ such that v_0 and w_0 have degree at least $\frac{n}{3}$. In this case, Maker in the first round claims an edge incident to v_0 which is not $\{v_0, w_0\}$, and in the second round claims an edge incident to w_0 which is not $\{v_0, w_0\}$ (this can be done since Breaker cannot claim all the edges incident to w_0 in two rounds). Thus Maker can claim two vertex-disjoint edges in this case.

Second, assume that $|A| = \frac{n}{2} + t$ for $t > 0$. Then all the vertices of $\overline{A}$ have degree at least $\lceil t + 1 \rceil \geq 2$ in $\mathcal{B}$. Maker starts by claiming an arbitrary edge $\{v, w\}$. Note that since all the vertices in $\overline{A}$ have degree at least 2, there are at least

$$|\overline{A}| - 1 \geq \left(\frac{1}{2} - 16\alpha\right)n - 1 \geq \frac{n}{3}$$

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edges remaining which are not incident to $\{v, w\}$. Breaker cannot claim all of these edges in two rounds, and thus Maker can claim one such edge in the next round to achieve his/her goal. This concludes the proof. $\qquad\square$

5.3 Third case

We assume that there exists a set A of size $\frac{n}{2} \leq |A| \leq (\frac{1}{2} + 16\alpha)n$ such that the bipartite graph induced by the edges between A and $\overline{A}$ has at least $(\frac{1}{4} - 14\alpha)n^2$ edges and minimum degree at least $\frac{n}{192}$. Moreover, either $|A| = \lceil \frac{n}{2} \rceil$, or the induced subgraph $G[A]$ has maximum degree at most $\frac{n}{96}$. Let $V_1 = A$ and $V_2 = \overline{A}$. Let $k = |V_1| - |V_2|$ so that $|V_1| = \frac{n+k}{2}$, $|V_2| = \frac{n-k}{2}$, and $k \leq 32\alpha n$.

Lemma 5.9. *There exist positive reals c and β such that the following holds for every $b \leq \frac{cn}{\log n}$. In a $(1:b)$ Maker-Breaker game played on G, Maker can construct a graph with property $\mathcal{RE}_b(\frac{1}{6})$ in the first $\beta n \log n$ rounds.*

Proof. Let β be a large enough positive constant. Let $\varepsilon = \min\{\frac{1}{150\beta}, \frac{1}{240}\}$ and $c \leq \frac{\varepsilon}{2\beta}$ be small enough. We start by splitting the board into two boards $G[V_1]$, and the bipartite graph induced by the edges between V_1 and V_2 (which we refer to as the *bipartite board $\mathcal{B}$*). We first show that on the board $G[V_1]$, Maker can claim at least $2k$ vertex disjoint edges. We only need to consider the cases when $k \geq 1$ since otherwise the claim is trivial. Suppose that Maker has claimed t vertex disjoint edges after t rounds for some $t < 2k$. Note that since $|V_2| = \frac{n-k}{2}$, the induced subgraph $G[V_1]$ has minimum degree at least $\frac{k}{2}$. Since the graph $G[V_1]$ has maximum degree at most $\frac{n}{96}$, the number of non-claimed edges not incident to any of the t edges that Maker has claimed so far is at least

$$\frac{1}{2} \cdot \frac{k}{2} \cdot \frac{n}{2} - 2t \cdot \frac{n}{96} - (t+1) \cdot \frac{cn}{\log n} \geq \frac{nk}{24}. \tag{5.3}$$

Maker chooses an arbitrary edge out of these edges. In the end, Maker can claim at least $2k$ vertex-disjoint edges in the board $G[V_1]$.

On the board $\mathcal{B}$, one can show that there exists a constant C which goes to infinity as c goes to zero, such that for $s = C \log n$, Maker can construct a graph satisfying the following:

1. For all $X \subset V_1$,

    (a) if $|X| \leq \frac{n}{s^{3/2}}$, then $|N(X) \cap V_2| \geq s|X|$,

    (b) if $\frac{n}{s^{3/4}} \leq |X| \leq \frac{n}{2^{30}}$, then $|N(X) \cap V_2| \geq \frac{n}{200}$,

    (c) if $\frac{n}{2^{30}} \leq |X|$, then $|N(X) \cap V_2| \geq \frac{7}{8}|V_2|$.

2. For all $Y \subset V_2$,

    (a) if $|Y| \leq \frac{n}{s^{3/2}}$, then $|N(Y) \cap V_1| \geq s|Y|$,

    (b) if $\frac{n}{s^{3/4}} \leq |Y| \leq \frac{n}{2^{30}}$, then $|N(Y) \cap V_1| \geq \frac{n}{200}$,

    (c) if $\frac{n}{2^{30}} \leq |Y|$, then $|N(Y) \cap V_1| \geq \frac{7}{8}|V_1|$.

To prove $1(a), 1(b), 2(a), 2(b)$, we can use the fact that the bipartite board has minimum degree at least $\frac{n}{192}$, and to prove $1(c), 2(c)$, we can use the fact that $e(A, \overline{A}) \geq (\frac{1}{4} - 14\alpha)n^2$. We omit the details which are similar to those of Lemmas 5.3 and 5.6.

It then suffices to show that one can carefully choose k of the edges within $G[V_1]$ as the special edges so that we can find a special frame, with respect to which Maker's graph is a bipartite-expander with certain parameters. Recall that Maker claimed at least $2k$ vertex-disjoint edges on the board $G[V_1]$. Call these *candidate edges*. Uniformly at random choose k edges among the candidate edges, and declare them as our special edges S_E. For each such edge, declare one of its vertices as a special vertex (let S_V be the set of special vertices). This forms a special frame (V_1, V_2, S_V, S_E) of Maker's graph. We now verify that Maker's graph is a bipartite-expander with parameters $\epsilon = \frac{1}{4}$ and $r = 2^{20}$ with respect to this frame.

Condition (i) of Definition 3.13 easily follows from 1(a), 1(b), and 1(c) given above. Let V_1'' be as in Definition 3.8. Fix a set $Y \subset V_2$ of size at most $\frac{n}{s^{3/2}}$, and note that $|N(Y) \cap V_1| \geq s|Y|$. We can find a subset N_Y of $N(Y) \cap V_1$ of size at least $\frac{s}{2}|Y|$ such that each candidate edge intersects N_Y in at most one vertex. Note that if $|N_Y \cap V(S_E)| < \frac{s}{3}|Y|$, then $|N(Y) \cap V_1''| \geq |N_Y \setminus V(S_E)| \geq \frac{s}{6}|Y|$. Since we randomly chose the special edges and the probability that each edge is chosen is at most $\frac{1}{2}$, the probability that $|N_Y \cap V(S_E)| \geq \frac{s}{3}|Y|$ is, by the concentration of hypergeometric distribution, at most $e^{-\Omega(s|Y|)}$. By taking the union bound over all possible choices of Y, we see that the probability that there exists a set Y for which $|N_Y \cap V(S_E)| \geq \frac{s}{3}|Y|$ is (for small enough c, and thus large enough C) at most

$$\sum_{i=1}^{n/s^{3/2}} \binom{n}{i} e^{-\Omega(is)} \leq \sum_{i=1}^{n/s^{3/2}} n^i n^{-\Omega(Ci)} \leq \sum_{i=1}^{n/s^{3/2}} n^{-i} = o(1).$$

Consequently, we can choose S_V and S_E so that $|N(Y) \cap V_1''| \geq \frac{s}{6}|Y|$ for all Y. Consider a set $Y \subset V_2$ of size between $\frac{n}{s^{3/2}}$ and $\frac{n}{s^{3/4}}$. For an arbitrary subset $Y' \subset Y$ of size $\frac{n}{s^{3/2}}$, as we explained above,

$$|N(Y) \cap V_1''| \geq |N(Y') \cap V_1''| \geq \frac{s}{6}|Y'| = \frac{n}{6s^{1/2}} \geq \frac{s^{1/4}}{6}|Y| > 2^{20}|Y|.$$

In the case $Y \subset V_2$ is of size between $\frac{n}{s^{3/4}}$ and $\frac{n}{2^{30}}$, we use the fact $|N(Y) \cap V_1| \geq \frac{n}{200}$. In this case, using that $\alpha = 2^{-40}$, we have that

$$|N(Y) \cap V_1''| \geq |N(Y) \cap V_1| - 2k \geq \frac{n}{200} - 64\alpha n \geq \frac{n}{300} \geq 2^{20}|Y|.$$

Therefore the first part of Condition (ii) of Definition 3.13 holds. To establish the second part, let $Y \subset V_2$ be a set of size at least $\frac{n}{2^{15}}$. For an arbitrary choice of S_E, we have

$$|N(Y) \cap V_1''| \geq |N(Y) \cap V_1| - 2k \geq \frac{7}{8}|V_1| - 64\alpha n \geq \frac{3}{4}|V_1|,$$

and this gives the second part of Condition (ii) of Definition 3.13 as well. Thus Maker's graph is a bipartite-expander with parameters $\frac{1}{4}$ and 2^{20}, and by Lemma 3.14, it follows that Maker's graph has property $\mathcal{RE}_b(\frac{1}{6})$. $\square$

Lemma 5.10. *For every positive real β, there exists a positive constant $c = c(\beta)$ such that the following holds for every $b \leq \frac{cn}{\log n}$. In a $(1 : b)$ Maker-Breaker game played on G, if Maker constructs a graph with property $\mathcal{RE}_b(\frac{1}{6})$ in the first $\beta n \log n$ rounds, then he can construct in the remaining rounds a graph that complements it.*

Proof. Let $c \leq \alpha \beta^{-1}$ be a small enough constant. Let M_1 be the Maker's graph constructed in the first $\beta n \log n$ rounds. Note that the number of edges that have been claimed in the first $\beta n \log n$ rounds is at most $c\beta n^2 + \beta n \log n \leq 2c\beta n^2$. Let G' be the graph induced by the edges that have not been claimed by Breaker so far. Let P be a proper path such that there is no path longer than P in the graph $P \cup M_1$. Then there exists a set $S_P \subset V_2$ of size $|S_P| \geq \frac{n}{6}$ such that for every $v \in S_P$ there exists a set $T_v \subset V_1$ of size $|T_v| \geq \frac{n}{6}$ such that for all $w \in T_v$, there exists a path of the same length as P starting at v and ending at w. By the fact $e(A, \overline{A}) \geq (\frac{1}{4} - 14\alpha)n^2 \geq |A| \cdot |\overline{A}| - 14\alpha n^2$, we have

$$e_{G'}(S_P, T_v) \geq |S_P| \cdot |T_v| - 14\alpha n^2 - 2c\beta n^2 \geq \frac{n^2}{36} - 16\alpha n^2 \geq \frac{n^2}{40}.$$

Using this estimate, we can proceed as in Lemma 5.4 to finish the proof. We omit the details. $\qquad\square$

Theorem 5.11. *There exists a constant c such that the following holds for every $b \leq \frac{cn}{\log n}$. If G is a Dirac graph satisfying (iii) of Lemma 2.1, then Maker has a winning strategy for the $(1:b)$-Maker-Breaker Hamiltonicity game.*

Proof. Let $\beta = \beta_{5.9}$, and $c = \min\{c_{5.9}, c_{5.10}(\beta)\}$. By Lemma 5.9, Maker can construct a graph with property $\mathcal{RE}_b(\frac{1}{6})$ in the first $\beta n \log n$ rounds. Then by Lemma 5.10, Maker can construct a graph which complements it in the remaining rounds. Therefore by Proposition 3.12, Maker can construct a Hamilton cycle and win the game. $\qquad\square$

6 Concluding Remarks

As we mentioned in the introduction, several measures of robustness of graphs with respect to various graph properties have already been considered before. In this paper, we propose two new measures, and strengthen Dirac's theorem according to these measures. Our first result asserts that there exists a constant C such that for $p \geq \frac{C \log n}{n}$ and an arbitrary Dirac graph G on n vertices, if one takes its edges independently at random with probability p, then the resulting graph is a.a.s. Hamiltonian. Our second theorem says that if one plays a $(1:b)$ Maker-Breaker game on a Dirac graph, then the critical bias for Maker's win is of order of magnitude $\frac{n}{\log n}$. We proved both of these theorems under one general framework.

It is worth comparing our results with two previous robustness results on Dirac graphs. Given a graph G, let $h(G)$ be the number of Hamilton cycles in G. Cuckler and Kahn [15], confirming a conjecture of Sárközy, Selkow, and Szemerédi [31], proved that $h(G) \geq \frac{n!}{(2+o(1))^n}$ holds for every Dirac graph G. Since the expected number of Hamilton cycles in the graph G_p is $h(G)p^n$, our first result which implies $h(G)p^n \geq 1$ for $p \geq \frac{C \log n}{n}$, recovers a slightly weaker inequality $h(G) \geq \left(\frac{n}{C \log n}\right)^n$. Another result of Lee and Sudakov [27] states that for $p \gg \frac{\log n}{n}$, every subgraph of $G(n, p)$ of minimum degree at least $\left(\frac{1}{2} + o(1)\right) np$ contains a Hamilton cycle. Even though there is no direct implication between that result and our result, they are nevertheless very closely related. Indeed, the result in [27] is similar in spirit to a slightly weaker version of our theorem, which says that for every fixed positive real ε and every graph G of minimum degree at least $\left(\frac{1}{2} + \varepsilon\right) n$, the graph G_p is a.a.s. Hamiltonian, since the resulting graph can be considered as a subgraph of $G(n, p)$ of minimum degree at least $\left(\frac{1}{2} + \frac{\varepsilon}{2}\right) np$.

We believe that two new measures of robustness that we proposed in this paper, taking a random subgraph, and playing a Maker-Breaker game, can be used to further study many other classical Extremal Graph and Hypergraph Theory results as well.

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